digits of a smaller number to that number. What is the largest of these finite number of integers? (*4 points*) **B. 5151.** Prove that if $a^2 = b^2 + ac = c^2 + ab$, then two of the numbers a, b, c are equal. (*3 points*) **B. 5152.** Determine those positive integers greater than 1 whose positive divisors can all be written on the circumference of a circle so that the ratio of every adjacent pair should be a prime. (*5 points*) (Proposed by *D. Lenger,* Budapest and *G. Szűcs,* Szikszó) **B. 5153.** Let A, B, C denote the vertices of an equilateral triangle of unit side, and let D be a point on the extension of side AB beyond B. The perpendicular drawn to line segment BC at B intersects line segment CD at E (see *figure*). Find the length of CE, given that $ED = 1$. (*4 points*) **B. 5154.** Find all functions f taking on positive integer values, and defined on the set of positive integers such that $f\big(f(n)\big) = 2n$ and $f(4n-3) = 4n-1$ for all positive integers n. (*4 points*) (Proposed by *S. Róka,* Nyíregyháza) **B. 5155.** The convex quadrilateral $ABCD$ has no parallel sides, and the intersection of lines AB and CD is M. Point X is moving along the interior of side AB, and point Y is moving along the interior of side CD so that the equality $AX : XB = DY : YC$ remains true. Show that the circles MXY all have another common point, different from M. (*5 points*) **B. 5156.** Let K be a convex polygon of $2n$ vertices where all sides have unit length and the opposite sides are parallel. Show that K can be dissected into a finite number of rhombuses of unit sides. How many rhombuses may there be? (*6 points*) **B. 5157.** There are some integers on the blackboard. Each of three students (independently of each other) selects a number from the board at random, and writes it down in their notebook. Prove that the probability that the sum of the three numbers written down is divisible by 3 is at least 1/4. (*6 points*)

New problems – competition A (see page 99): **A. 793.** In the 43 dimension Euclidean space the convex hull of finite set S contains polyhedron P. We know that P has 47 vertices. Prove that it is possible to choose at most 2021 points in S such that the convex hull of these points also contain P, and this is sharp. (Submitted by *Dömötör Pálvölgyi,* Budapest) **A. 794.** A polyomino P occupies n cells of an infinite grid of unit squares. In each move, we lift P off the grid and then we place it back into a new position, possibly rotated and reflected, so that the preceding and the new position have $n-1$ cells in common. We say that P is a caterpillar of area n if, by means of a series of moves, we can free up all cells initially occupied by P. How many caterpillars of area 10^6+1 are there? (Submitted by *Nikolai Beluhov,* Bulgaria)

Problems in Physics

(see page 121)

M. 402. Make an 80-cm long paper strip from a thin sheet of paper, and attach its ends at the same height to two movable stands, which are at a certain distance from each other. Place a small size cylinder-shaped unopened tin can on the strip as shown in the *figure,* and let it roll without initial speed. In the case of different distances of d between the stands measure the greatest speed of the symmetry axis of the can. At what distance of d will this speed be the maximum?

G. 733. Water is drawn up from a well. The depth of the well is 10 metres, the mass of the bucket is 2 kg, the mass of the chain is 3 kg and the capacity of the container is 12 litres. What is the mechanical efficiency of bringing the water up? Does this efficiency depend on the depth of the well? **G. 734.** A car travelled for 3 hours at an average speed of 80 km/h. The route consisted of three parts in all of which the car travelled at a constant speed. In the heavy traffic of some urban area its speed was $v_1 = 20$ km/h, on the highway its speed was $v_2 = 80$ km/h, while on the dual carriageway its speed was $v_3 = 120$ km/h for one and a half hours. How long was the car in the heavy traffic of the city, and what was the average speed of the car for the total distance travelled in the city and on the highway? **G. 735.** The pulley wheel shown in the *figure* can rotate freely about a fixed shaft. The masses of the movable pulleys are m_1 and m_2 ($m_1 < m_2$). In what direction and at what magnitude of force do we have to pull the thread at the left in order to keep the system in equilibrium? (The thread does not slide on the pulley.) *Data:* $R = 10$ cm, $r = 5$ cm, $m_1 = 2$ kg, $m_2 = 3$ kg. **G. 736.** A big bowl full of water extends beyond the edge of the table so far, that it is just *on the verge* of tipping over. A piece of ice cube is floating in the water in that part of the bowl which is above the table. A very gentle breeze wafts it towards the part of the bowl which is not above the table. When will the bowl tip over?

P. 5294. A half-cylinder-shaped trough has a horizontal symmetry axis. Through the midpoint of one of the horizontal radii of the trough an inclined plane of angle of elevation of α is laid. What is the angle of elevation of that slope along which a small object slides without friction in the shortest time? **P. 5295.** A small cart of mass m which is driven by a LEGO motor starts to move upward along a slope of angle of elevation of α. The mechanical power of the motor (except for the very beginning of the motion) has a constant value of P. What is the final speed of the cart? (The wheels do not slide and rolling resistance is negligible.) *a*) Describe the motion of the cart. *b*) As a function of the time sketch the graph of the power, speed, and the static frictional force in the same diagram. Sketch the force–velocity diagram as well. *c*) What is the least frictional force during the motion? **P. 5296.** A hollow iron ball is floating in water which is at a temperature of 1 °C. What happens if the temperature slowly rises? What is the temperature of the water in which the ball floats again? **P. 5297.** The length of a light, flexible but unstretchable fishing line is $\ell = 80$ cm. The two ends of the fishing line are fixed at two points at the same height, at a certain distance of each other. A steel marble of mass $m = 5$ g, which has a hole drilled through it, can slide along the fishing line. This bead is started from a position such that one part of the tight fishing line is vertical. *a*) To what maximum speed can the steel marble speed up, if friction and air drag are negligible? *b*) What is the tension in the fishing line when the speed of the steel marble is maximum? **P. 5298.** Two small balls each having a mass of $m = 0.25$ kg are tied by means of an $\ell = 60$ cm long thread, which cannot be stretched. The balls are held such that the thread is horizontal and the tension is zero. At a certain moment the two balls are released, without jerking any of them. After a fall of $h = 1.8$ m one of the balls collides with a protruding rigid stone ledge. The collision is totally elastic. *a*) What is the tension in the thread at the moment right after the collision? *b*) How high above the ledge will the ball, colliding with the ledge, be when $t = 0.25$ s elapses after the collision? (Air drag is negligible.) **P. 5299.** A strong uniform-density spring of mass M and of spring constant D is accelerated by pulling it at one of its ends with a force F along a horizontal, frictionless tabletop such that each point of the spring moves at the same acceleration into the direction of the symmetry axis of the spring. What is the length of the spring if its relaxed length was $\ell_0 \gg F/D$? **P. 5300.** The principal axis of a lens of diameter 10 cm, and of focal length 20 cm points exactly towards the centre of the Sun. A plane mirror is placed 30 cm from the lens as it is shown in the *figure.* Where should a very small object be placed on the principal axis in order that its temperature increase at the greatest rate? Assuming clear sunny weather how long would it take for a small piece of ice placed to that position to melt provided that the ice is kept in a small aluminium jar covered in black soot? The volume of the ice is 0.1 cm^3 and it is at a temperature of 0 °C. The total energy lost in the lens, in the mirror and on the surface of the bodies, is the 20% of the useful energy. The intensity of the radiation of the Sun at the surface of the Earth is 0.1 $\mathrm{W/cm}^2$, and the image of the Sun is smaller than the size of the jar. Along the principal axis there is another point at which the ice in the jar melts sooner than at the neighbouring points. Where is this point and how long does it take for the ice to melt there? **P. 5301.** In the electric field of a point-like charge Q, at a distance of R from it, there is a point-like electric dipole, which can rotate freely and which has a dipole momentum of p. How much work has to be done when the dipole is moved very far ("to infinity") from the charge? **P. 5302.** A tube, closed at its bottom end and open at the its top, having a cross sectional area of A, is held vertically as it is shown in the *figure* such that at its top we also hold a heavy, solid piston of mass m, which can move easily in the tube and which extends a bit beyond the tube. Initially the external pressure p_0 is the same as the pressure inside the tube. The initial length of the air column in the tube is L. The bottom of the tube is at a distance of h_0 from the level ground. The ambient temperature and the initial temperature inside is T_0. The system is released without initial speed at a certain moment. The bottom of the tube sticks to the ground when it collides with it. (Friction inside the tube and the external air resistance is negligible.) *a*) What will the maximum temperature inside the tube be? *b*) What will the maximum acceleration of the piston be? *c*) To what height will the piston go up after leaving the tube? *Data:* $A = 0.25\ \mathrm{dm}^2$, $m = 0.5$ kg, $p_0 = 10^5$ Pa, $L = 0.8$ m, $h_0 = 0.6$ m, $T_0 = 300$ K. **P. 5303.** The bullet of a gun hits a wooden target at a speed of 500 m/s, and penetrates into it to a depth of 5 cm. The bullet can be considered as a solid cylinder of length 4 cm, and of density 7800 $\mathrm{kg/m}^3$, which is decelerated uniformly. *a*) Estimate the maximum of the mechanical tension occurring due to the deceleration of the bullet. *b*) Estimate the maximum value of the voltage which can be measured between the two ends of the cylinder due to the inertia of the electrons. **P. 5304.** A stationary body is located at the equator. In which case will the apparent weight of the object be smaller: at noon or at midnight? What is the relative change in the apparent weight of the object in 12 hours? Neglect the effect of any celestial bodies other than the Sun and the Earth.

Problems of the 2020 Kürschák competition

1. Let n, k be positive integers. Suppose that we have n closed discs in the plane such that among any $k+1$ of them there are two with no common point. Prove that it is possible to divide the discs into at most $10k$ classes with the property that two discs from the same class never have a common point.

2. Give the functions f which map the set of rational numbers to the set of nonnegative real numbers and satisfy the conditions

- $f(x+y) \leqslant f(x) + f(y)$ for any rational numbers x, y,
- $f(xy) = f(x)f(y)$ for any rational numbers x, y,
- $f(2) = 1/2$.

3. There are N houses in a city. Each Christmas Santa visits the houses in a certain order. Prove that if N is sufficiently large, then for three consecutive years it is always possible to choose 13 houses that were visited by Santa in the same order in at least two (out of the three) years. Determine the smallest N for which this implication holds.