P. 5293. Egy feketedoboz tetején sok kivezetés van. Tudjuk, hogy belül minden kivezetéspár közé egy-egy ismeretlen ellenállást forrasztottak. Hogyan mérhetjük meg két tetszőleges pont közé kötött ellenállás értékét, ha csupán ellenállásmérőnk és tetszőleges számú röpzsinórunk van?

(*6 pont*) Közli: *Vladár Károly,* Kiskunhalas

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Problems in Mathematics

New exercises for practice – competition K (see page 31): **K. 679.** Pete was three years old when he got a set of six rectangular building blocks. The dimensions of the blocks are 1 dm × 1 dm × 2 dm. The dimensions of the interior of the box for storing the blocks are 3 dm × 2 dm × 2 dm and each face is different in colour. In how many different arrangements may Pete place the blocks in the box if the blocks are identical in colour and cannot be distinguished? (It is not allowed for any block to stick out of the box). **K. 680.** Four faces of a cube are coloured red, and then the cube is cut into 125 identical small cubes. What may be the number of small cubes with no red face? **K. 681.** How many triangles are there in which the sides have integer lengths in centimetres, and the longest side is 2021 cm long? (There may be more than one side of this length.) **K. 682.** There is a sufficient number of copies of three different cards, with one digit on each. All possible four-digit positive numbers are formed out of the cards. The sum of these numbers is 689 931. What are the three digits on the cards? **K. 683.** A heptagon $ABCDEFG$ is inscribed in a circle. The sum of angles $\angle ABC$, $\angle CDE$ and $\angle EFG$ is greatrer than 450°. Show that the centre of the circumscribed circle cannot lie either inside the heptagon or on its boundary. (*The University of Stirling, school mathematics competition, 1983*)

New exercises for practice – competition C (see page 32): **Exercises up to grade 10: C. 1644.** We have made a 10 cm × 30 cm rectangular tin of shortbread. It has a delicious crispy edge. We want to divide the bread into pieces by using cuts running all the way parallel to the edges of the tin. How many pieces may we obtain if we would like each piece to have the same length of crispy edge? **C. 1645.** In an acute-angled triangle, the sides are a, b, c, and m_b is the height drawn to side b. The lengths of m_b, a, b, c, in this order, are consecutive positive integers. What is the area of the triangle? (Proposed by *Zs. M. Tatár,* Esztergom) **Exercises for everyone: C. 1646.** Find the integer solutions of the equation $(xy-1)^2=(x+1)^2+(y+1)^2$. (Proposed by *M. Szalai,* Szeged) **C. 1647.** The medians drawn to the legs of an isosceles triangle are perpendicular to each other. Let r and R denote the inradius and circumradius, respectively. Find the exact value of the ratio $\frac{r}{R}$. **C. 1648.** King Arthur and Sir Lancelot are running a horse race. Sir Lancelot says, “Since Your Majesty’s speed is only $\frac{2}{3}$ of mine, I will give Your Majesty a handicap of 100 metres, and then I would catch up within the length of the race track. Alternatively, if Your Majesty reduced speed by 2 $\frac{\mathrm{m}}{\mathrm{s}}$ and I reduced mine by

$5\ \frac{\mathrm{m}}{\mathrm{s}}$, but I gave Your Majesty a 50 metre handicap only, I would also be able to catch up within the length of the track. The sum of the two time intervals required to catch up is exactly 75 seconds." Determine the speeds of King Arthur and of Sir Lancelot. **Exercises upwards of grade 11: C. 1649.** The diagonals of a cyclic quadrilateral intersect each other at right angles at point M. The diagonals divide the quadrilateral into triangles. Prove that the altitude of any triangle drawn from point M is collinear with the median drawn from point M in the opposite triangle. **C. 1650.** Prove the inequality $\log_{ab} c \leqslant \frac{\log_a c+\log_b c}{4}$ for $a, b, c > 1$.

New exercises – competition B (see page 33): **B. 5142.** In a football championship, there are four teams in a group. Within the group, each team plays every other team once. The teams receive 3 points for winning, 1 point for a draw and 0 points for losing a game. The two teams scoring the highest qualify for the semi-finals, and the other two teams are eliminated. In the case of equal scores, the qualification is decided by chance. Determine those values of the number p for which it may happen that a qualifying team and an eliminated team both have p points. (*3 points*) **B. 5143.** Find the real solutions of the equation $16x^2 + 9x + 117 = 24x\sqrt{x+13}$. (*4 points*) (Proposed by *S. Róka,* Nyíregyháza) **B. 5144.** The area of a convex quadrilateral $ABCD$ is t, and an interior point is O. Show that $2t \leqslant OA^2 + OB^2 + OC^2 + OD^2$. When will equality occur? (*3 points*) **B. 5145.** Show that there are $\binom{n+1}{2k+1}$ different strings of zeros and ones of length n in which it occurs exactly k times that a 0 is followed by a 1. (*4 points*) (*Problem from a qualifying competition in England for the Olympiad*) **B. 5146.** T is a cuboid of unit volume, and M is a point in its interior. Point M is reflected in the planes of the faces. Let D be the convex hull of the 6 images obtained. Determine the volume of the solid $T \cap D$. (*5 points*) **B. 5147.** Let $k > 1$ be a positive integer. Is there *a*) a finite subset (of any size) *b*) an infinite subset of the set of positive integers in which the greatest common divisor of any k elements is greater than 1 but the greatest common divisor of any $k+1$ elements is equal to 1? (*5 points*) (Proposed by *G. Mészáros,* Budapest) **B. 5148.** A triangle ABC is right angled at C. The inscribed circle touches the leg BC at point D, and the leg AC at point E. The escribed circle of side BC touches line segment BC at point G; and the escribed circle of side AC touches line segment AC at point H. The intersection of line segments DH and EG is M. Show that the other intersection of the circumscribed circles of triangles DGM and EHM lies on the inscribed circle. (*6 points*) **B. 5149.** In how many different ways is it possible to fill in a 6×6 table with the numbers $1, 2, \ldots, 36$ so that however 6 fields are selected, all lying in different rows and in different columns, the sum of the numbers in 6 such fields should always be the same? (*6 points*)

New problems – competition A (see page 34): **A. 791.** A lightbulb is given that emits red, green or blue light and an infinite set S of switches, each with three positions labeled red, green and blue. We know the following: *i*) For every combination of the switches the lighbulb emits a given color. *ii*) If all switches are in a position with a given color, the lightbulb emits the same color. *iii*) If there are two combinations of the switches where each switch is in a different position, the lightbulb emits a different color for the two combinations. We create the following set U containing some of the subsets of S: for each combination of the switches let us observe the color of the lightbulb, and put the set of those switches in U which are in the same position as the color of the lightbulb. Prove that U is an ultrafilter on S. (U is an ultrafilter on S if it satisfies the following: *a*) The empty set is not in U. *b*) If two sets are in U, their intersection is also in U. *c*) If a set is in U, every subset of S containing it are also in U. *d*) Considering a set and its complement in S, exactly one of these sets is contained in U.) See also problem **N. 35.**[*] from the May issue of 1994 (in Hungarian). **A. 792.** Let $p \geqslant 3$ be a prime number and $0 \leqslant r \leqslant p-3$. Let $x_1, x_2, \ldots, x_{p-1+r}$ be integer numbers satisfying $\sum_{j=1}^{p-1+r} x_j^k \equiv r \pmod{p}$ for all $1 \leqslant k \leqslant p-2$. What are the possible remainders of numbers $x_1, x_2, \ldots, x_{p-1+r}$ modulo p? (Submitted by *Dávid Matolcsi,* Budapest)

Problems in Physics

(see page 57)

M. 401. Make a physical pendulum of mass m and of length ℓ, from a uniform-density thin wooden slat which is pivoted at one of the ends. (The values of m and ℓ can be chosen arbitrary, but should be kept constant during the measurement.) *a*) Measure the period of the pendulum T_0 after it is displaced a bit. Then change the position of the pivot by positioning it at a distance of d from one of the ends of the rod, and attach a point-like object of mass M, for example a small piece of plasticine, to the other end. If the mass of the plasticine is chosen carefully, then the period of this pendulum is the same as the original period T_0. *b*) Measure how the ratio of the masses M/m depends on the ratio of the distances d/ℓ.

G. 729. When melted lard is left to cool down in a pot, it can be observed clearly that the surface of the lard is similar to a crater, along the rim a regular flange is formed. Why? **G. 730.** In a bicycle race the first and the second riders are cycling at a constant speed of $v_0 = 50$ km/h. The first rider is 100 m ahead of the second. At a certain moment —close to the finish— the third cyclist begins to speed up and overtakes the second rider at a speed of $v_1 = 55$ km/h, and he is able to maintain this speed. How far is the finish from the point where the overtaking occurred if the first cyclist wins the race? **G. 731.** In a suburban area, where the the speed limit is 30 km/h, a car —a bit illegally— travels at a speed of 36 km/h. Another similar car overtakes it at a speed of 54 km/h. They are just next to each other when a child, who is 20 m ahead, runs to the road. Both drivers start to brake at the same moment, pushing the brakes at the same force. *a*) At what “remaining” speed does the faster car pass the child, if the other car just stops in front of the child? *b*) How does the result change if we consider that both drivers’ reaction time is approximately 1 second? **G. 732.** News report (November 17, 2020): “The Crew Dragon spacecraft has arrived at the International Space Station (ISS). After a 27-hour totally autonomous flight it docked with the station, which was floating at a height of approximately 400 kilometres above the surface of the Earth.” Estimate the following. *a*) How many times did the spaceship go around the Earth from its launch until it docked? *b*) What was the speed of the “floating” space station when the spaceship docked with it?

P. 5283. Three friends Sebi, Tóni and Zoli entered for the school’s running competition held on Challenge Day. All of them covered the 2.4 km distance at a constant speed. When Tóni just covered 68% of the distance, Sebi had another three minutes to run. Zoli covered 20 cm more in each second than Sebi did, while he covered 10 cm less in each second than Tóni did. *a*) How much time elapsed between the moments when Zoli and Tóni reached the finish line? *b*) How far was Sebi from the finish line when Tóni reached it? **P. 5284.** The following can be read on the bottle of an alcoholic disinfectant solution: “Active ingredients: ethyl alcohol (70 V/V%)”. The active ingredient contained by another type of solution is 67.9 m/m% ethanol (ethyl alcohol). Assuming that the amount of other additives is negligible, which solution has greater alcohol concentration? (The density of the ethanol-water mixture as a function of concentration can be found in tables.) Give a generally applicable relationship between the concentration of the solution, expressed

[*] `http://db.komal.hu/KomalHU/showpdf.phtml?tabla=FelHivatkoz&id=41643`