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Problems in Mathematics

New exercises for practice – competition C (see page 288): **Exercises up to grade 10: C. 1546.** Find the integer solutions of the equation $(x-8)(x-10)=2^y$. (*American competition problem*) **C. 1547.** Let K denote the midpoint of side EF in a regular hexagon $ABCDEF$. Find the point L on the broken line $ABCD$ for which the area of triangle AKL is $\frac{2}{5}$ of the area of the hexagon. (Based on a problem by *T. Bakos*) **Exercises for everyone: C. 1548.** Ann selects a few fields of a 3×3 table. Then row by row and column by column, she tells Bill how many selected fields there are in that row or column. In how many different ways may Ann select her fields so that Bill cannot find out from the given information which fields she selected? **C. 1549.** Let F be the midpoint of a line segment AB, and let Z be an arbitrary point on line segment AF. Draw a perpendicular to AB at F, and mark a distance $FX=FA$ on it. Draw another perpendicular to AB at B, and mark a distance $BY=AZ$ on it such that X and Y lie on the same side of line AB. What may be the size of the angle XZY? (Proposed by *L. Surányi,* Budapest) **C. 1550.** Find all positive integers n satisfying $n\cdot(1!+2!+3!+\dots+n!)=(n+1)!$. **Exercises upwards of grade 11: C. 1551.** In a triangle ABC, the lengths of medians AD and BE are 3 cm and 6 cm, respectively, and the area of the triangle is $3\sqrt{15}\,\mathrm{cm}^2$. Determine the length of the third median, given that it is different from the other two. **C. 1552.** Prove that if $0<a<1$ and $0<b<1$ then $\log_a \frac{2ab}{a+b}\cdot\log_b\frac{2ab}{a+b}\geqslant 1$. (Proposed by *S. Róka,* Nyíregyháza)

New exercises – competition B (see page 289): **B. 5030.** Prove that every integer greater than 1 can be represented as a sum of numbers of the form $2^p\cdot 3^q$ greater than 1 such that no term of the sum is a divisor of another term. (For example, $23=9+8+6$, $11=9+2$ or $12=12$.) (*4 points*) (A problem of *Paul Erdős*) **B. 5031.** Let F be a point on the extension of side AD of parallelogram $ABCD$ beyond vertex D. Line segment BF intersects side CD at G and diagonal AC at E. Show that $\frac{1}{BE}=\frac{1}{BG}+\frac{1}{BF}$. (*3 points*) **B. 5032.** In the interior of an isosceles triangle, what is the locus of points for which the distance from the base is the geometric mean of the distances from the legs? (*4 points*) **B. 5033.** The $\binom{n+1}{2}$ numbers $a_{1,1}, a_{1,2}, \dots, a_{1,n}, a_{2,1}, a_{2,2}, \dots, a_{2,n-1}, \dots, a_{k,1}, \dots, a_{k,n+1-k}, \dots, a_{n,1}$ are called an inverted Pascal pyramid of order n if $a_{k,j}=a_{k-1,j}+a_{k-1,j+1}$ for an arbitrary $2\leqslant k\leqslant n$ and $1\leqslant j\leqslant n+1-k$. An example of an inverted Pascal pyramid of order 3 is shown below:

$$\begin{array}{ccccc} & & a_{3,1}=2 & & \\ & a_{2,1}=1 & & a_{2,2}=1 & \\ a_{1,1}=-2 & & a_{1,2}=3 & & a_{1,3}=-2 \end{array}$$

Let s_k denote the sum of the numbers in row k of the pyramid, that is, $s_k=a_{k,1}+a_{k,2}+\dots+a_{k,n+1-k}$. A pyramid is said to have a sign change in row k ($k>1$) if $s_{k-1}\cdot s_k<0$. Given the value of n, what is the largest possible k if a pyramid of order n has sign changes in rows 2, 3, …, k, but no sign change in row $(k+1)$? (In the example above, $k=2$ since $s_1\cdot s_2=-2<0$ but $s_2\cdot s_3=4>0$.) (*5 points*) **B. 5034.** Prove that if none of the angles α, β, γ, δ of a convex quadrilateral are right angles then $\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=\tan\alpha\cdot\tan\beta\cdot\tan\gamma\cdot\tan\delta(\cot\alpha+\cot\beta+\cot\gamma+\cot\delta)$. (*3 points*) (A problem of *J. Surányi*) **B. 5035.** The edges of a complete graph on $n\geqslant 8$ vertices are coloured in two colours. Prove that the number of cycles formed by four edges of the same colour is more than

$\frac{(n-5)^4}{64}$. (*6 points*) (Based on a problem proposed by *M. Pálfi,* Budapest) **B. 5036.** From a point M, two tangents are drawn to a right-angled hyperbola of centre O. One tangent intersects an asymptote at point P, and the other tangent intersects the other asymptote at point Q. Prove that line OM bisects the line segment PQ. (*5 points*) **B. 5037.** A polyhedron P is divided into the smaller polyhedra $P_1, \ldots, P_k$ and also divided into the smaller polyhedra $Q_1, \ldots, Q_k$ such that the polyhedra P_i and Q_i are congruent for all $i = 1, \ldots, k$. Show that it is possible to select some points in the interior of P such that there are the same number of points (at least one) in the interior of polyhedra P_i and Q_i for all $i = 1, \ldots, k$. (The position of each polyhedron is fixed in the space.) (*6 points*)

New problems – competition A (see page 291): **A. 752.** Let k, s and n be positive integers such that $s < (2k+1)^2$, and let R be the set of lattice points (x, y) in the plane, satisfying $1 \leqslant x, y \leqslant n$. On the grid R we perform the following procedure. Initially, we colour one point of R green; all other points in R are coloured white. On every move, we choose a square S, consisting of $2k \times 2k$ lattice points in such a way that the center of S is green and it contains at least s white points; then we re-colour s white points of S to green. We repeat this step as long as there is a suitable square S. We say that the number s is *k-sparse*, if there exists a positive real number C such that, for every n, for every choice of the initial green point, and for every valid sequence of moves, the total number of green points in the grid cannot exceed Cn. Determine the least k-sparse positive integer s in terms of k. (Proposed by: *Nikolai Beluhov,* Stara Zagora, Bulgaria) **A. 753** Let a be an integer, and let p be a prime divisor of $a^3 + a^2 - 4a + 1$. Show that there is an integer b such that $p \equiv b^3 \pmod{13}$. **A. 754** Let P be a point inside the acute triangle ABC, and let Q be the isogonal conjugate of P. Let L, M and N be the midpoints of the shorter arcs BC, CA and AB of the circumcircle of ABC, respectively. Let X_A be the intersection of ray LQ and circle PBC, let X_B be the intersection of ray MQ and circle PCA, and let X_C be the intersection of ray NQ and circle PAB. Prove that P, X_A, X_B and X_C are concyclic or coincide. (Proposed by: *Gustavo Cruz,* São Paulo)

Problems in Physics

(see page 312)

M. 387. Cut an approximately spherical orange into two parts and place one of the two “hemispheres” on a slope, with the curved part of the orange touching the surface of the slope. Use a slope whose angle of elevation can easily be changed, and the surface of which is rough enough so the orange does not slip on it. Increase the angle of elevation of the slope, until the orange stays at rest in a slant position. Take a picture of the orange on the slope. Measure the maximum angle of elevation of the slope, and construct the centre of mass of the hemisphere.

G. 673. A cuboid-shaped aquarium is slowly filled with water. By what factor the force exerted on a wall of the cuboid is greater when the aquarium is fully filled with water than when the aquarium is filled only to one third of its height? **G. 674.** Between Budapest and Veresegyház there are two types of trains: passenger trains and fast passenger trains. Determine the average speeds of both types of trains using a railway timetable available on the internet (for example `elvira.mav-start.hu`). How will the average speeds of the trains change if the train has to wait ten minutes for another train coming from the opposite direction? **G. 675.** A plane mirror is placed horizontally on a horizontal floor, and above it another plane mirror, facing towards the first one is placed as well. There is a small black spot in the middle of the top mirror. The mirror at the top is released and begins to fall at an acceleration of g. What are the magnitudes and the directions of the acceleration of the images of the spot? **G. 676.** At dawn on 21 January 2019 there