*a*) Mekkora a csillag tengelyforgási periódusa, ha az átmérője $1{,}4\cdot 10^9$ m? Tételezzük fel, hogy a csillag forgástengelye merőleges a látóirányunkra, és a vonalkiszélesedést főként a csillag forgása okozza.

*b*) Milyen következtetést vonhatnánk le a csillag mozgásáról, ha a vonalat 410,176 nm és 410,182 nm közötti tartományra kiszélesedve észlelnénk?

(*5 pont*) Közli: *Kovács József,* Szombathely

**P. 5099.** Egy hullámvasút kocsija egy függőleges síkban fekvő, kör alakú pályán halad úgy, hogy a saját motorját és fékjét használva a sebességét állandó értéken tartja. Legalább mekkora sebességet kell tartania ahhoz, hogy az $R$ sugarú pályán megcsúszás nélkül tudjon végighaladni, ha a tapadó súrlódás együtthatója $\mu$? Hol csúszna meg, ha a sebessége ennél kicsit kisebb lenne? A kocsi elég kicsi a pálya sugarához képest.

(*6 pont*) Közli: *Takács László,* Baltimore, USA

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## Problems in Mathematics

**New exercises for practice – competition K** (see page 32): **K. 609.** Given that 50 minutes ago the time past 3 p.m. was four times as many minutes as the time to 6 p.m. (in the same afternoon), what time is it now? **K. 610.** We are building a flight of stairs from solid concrete. It leads to a height of 3 metres, and its width is 1 metre. Each step has a height ($m$) and a so-called depth ($l$), as shown in the *figure.* It is required that $2m+l=64$ cm, and a stair is not allowed to have a height greater than its depth. What is the minimum possible number of steps? How much concrete is needed for the flight of stairs that has the minimum number of steps? **K. 611.** Is it possible to arrange the integers 1 to 50 in pairs such that the sums of the numbers in the pairs are all distinct primes? **K. 612.** Find all positive integers $n$ for which $n+125$ and $n+201$ are perfect squares. **K. 613.** Two persons are playing the following game: they take turns in writing positive integers not greater than 10 on a blackboard. A number is not allowed if it is a factor of some number that has been written before. The player who is not able to write a new number on the board will lose the game. Which player has a winning strategy? (Proposed by *L. Loránt*)

**New exercises for practice – competition C** (see page 33): **Exercises up to grade 10: C. 1518.** How many 13-digit positive integers are there which contain only digits of 3, 6, 9, and in which the difference between every pair of consecutive digits is 3? **C. 1519.** The lengths of two sides of a triangle are 31 and 22. The medians drawn to these sides are perpendicular. How long is the third side? **Exercises for everyone: C. 1520.** Determine the last two digits of the number $2^{2019}+2019^2$. **C. 1521.** A circle of half the radius touches a circle of centre $O$ from the inside at point $E$. A ray drawn from $O$ intersects the large circle at $P$, and the other intersection with the small circle is $R$. Prove that the arcs $\widehat{EP}$ and $\widehat{ER}$ have the same length. **C. 1522.** The positive integers are arranged in three rows as follows:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 7 | 10 | 13 | 16... |
| 2 | 5 | 8 | 11 | 14 | 17... |
| 3 | 6 | 9 | 12 | 15 | 18... |

Prove that it is possible to select an infinite geometric sequence out of each row. **Exercises upwards of grade 11: C. 1523.** A convex quadrilateral is divided into triangles by its diagonals. Prove that if there are exactly three different values among the areas of the four triangles then the quadrilateral is a trapezium. **C. 1524.** Let $N$ and $M$ be positive integers, and let $p$ and $q$ denote different primes. Assume that $N+M$ is a five-digit number, $N$ is divisible by $p$ and the number of its divisors is $q$, while $M$ is divisible by $q$ and the number of its divisors is $p$. Determine all possible values of $N$ and $M$.

**New exercises – competition B** (see page 34): **B. 4998.** A set of small plastic figures used as a visual aid in primary-school mathematics consists of 48 pieces. The figures can be classified according to four different attributes: their size may be large or small, they may have or not have a hole in the middle, their colour may be red, yellow, blue or green, and their shape may be circular, square or triangular. Every possible combination of the four properties (e.g. small, blue circular disc with a hole) occurs exactly once. How many pieces $x$ are there in the set, such that there exists a piece $y$ for which the two conditions below both hold? 1. If $x$ is red or has no hole, then $y$ is a small yellow square. 2. If $y$ is small or blue, then $x$ is a green triangle or some figure without a hole. (*4 points*) (Based on a calculus test problem for freshmen at ELTE, Budapest) **B. 4999.** The incentre of a triangle $ABC$ is $O$, the points of tangency on the sides are $A_1$, $B_1$, $C_1$. The points of tangency of the excircles are $A_2$, $B_2$, $C_2$ as shown in the *figure*. Show that the area of one of the triangles $OA_1A_2$, $OB_1B_2$ and $OC_1C_2$ equals the sum of the areas of the other two triangles. (*3 points*) (Proposed by: *Sz. Kocsis,* Budapest) **B. 5000.** One of 4999 distinct given integers is 42. Prove that it is possible to select some numbers such that their sum is divisible by 5000. (*4 points*) **B. 5001.** The base of an isosceles triangle is $a$, its apex angle is smaller than $120^\circ$, and the altitude drawn to the base is $m$. Each vertex of the triangle is reflected in the line of the opposite side. The three reflections form a new isosceles triangle, with a base $a'$ and an altitude $m'$ drawn to the base. Show that $\frac{a'}{a}+\frac{m'}{m}=4$. (*3 points*) (Proposed by *P. Bártfai,* Budapest) **B. 5002.** The graph of the cubic polynomial $x^3+ax^2+bx+c$ and the circle of radius 10 centred at the origin intersect at six distinct points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$. Express the centre of mass of the system $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$ in terms of the coefficients $a$, $b$, $c$. (*5 points*) **B. 5003.** Is it true that if five out of the midpoints of the six edges of a tetrahedron lie on the same sphere, then the sixth midpoint also lies on that sphere? (*5 points*) **B. 5004.** In a set of $2n$ consecutive integers, what is the maximum possible number of elements that are divisible by at least one of the numbers $n+1$, $n+2$, $\dots$, $2n$? (*6 points*) (Proposed by *S. Róka,* Nyíregyháza) **B. 5005.** The feet of the altitudes of an acute-angled triangle $ABC$ drawn to the sides $BC$, $CA$, $AB$ are $D$, $E$, $F$, respectively. The orthocentre of triangle $ABC$ is $M$. Let $k_1$ denote the circle of diameter $AB$, and let $k_2$ be the circumscribed circle of triangle $DEM$. Let $P$ be an interior point of the arc $EM$ of $k_2$ that does not contain point $D$. Let line $DP$ intersect circle $k_1$ again at point $Q$, and let the midpoint of line segment $PQ$ be $R$. Show that lines $AQ$, $MP$, $FR$ are concurrent. (*6 points*) (Proposed by *B. Bíró,* Eger)

**New problems – competition A** (see page 35): Our problem number **A. 738.** was appeared wrongly in December. The solution of the new problem can be accepted together with the January problems. **A. 738.** Consider the following sequence: $a_1=1$,

$a_2=2$, $a_3=3$, and $a_{n+3}=\frac{a_{n+1}^2+a_{n+2}^2-2}{a_n}$ for all integers $n\geqslant 1$. Prove that every term of the sequence is a positive integer. **A. 740.** A $k\times k$ array contains each of the numbers $1,2,\dots,m$ exactly once, with the remaining entries all zero. Suppose that all the row sums and column sums are equal. What is the smallest possible value of $m$ if $k=3^n$ ($n\in\mathbb{N}^+$)? (Proposed by: *Attila Sztranyák* and *Péter Erben,* based on a problem of the 2017 Kalmár competition) **A. 741.** Let $f$ be a function defined on the positive integers with $f(n)\geqslant 0$ and $f(n)\leqslant f(n+1)$ for all $n$. Prove that if $\sum_{n=1}^{\infty}\frac{f(n)}{n^2}$ diverges, there exists a sequence $a_1,a_2,\dots$ such that the sequence $\frac{a_n}{n}$ hits every rational number, while $a_{n+m}\leqslant a_n+a_m+f(n+m)$ holds for every pair $n,m$. (Based on a problem of the Miklós Schweitzer competition) **A. 742.** Convex quadrilateral $ABCD$ is inscribed in circle $\Omega$. Its sides $AD$ and $BC$ intersect at point $E$. Let $M$ and $N$ be the midpoints of the circle arcs $AB$ and $CD$ not containing the other vertices, and let $I$, $J$, $K$, $L$ denote the incentres of triangles $ABD$, $ABC$, $BCD$, $CDA$, respectively. Suppose $\Omega$ intersects circles $IJM$ and $KLN$ for the second time at points $U\neq M$ and $V\neq N$. Show that the points $E$, $U$, and $V$ are collinear.

### Problems in Physics

(see page 58)

**M. 383.** Place a sand-glass on a slope, and measure how long it takes for the sand to run down through the hole of the sand-glass as a function of the angle of the slope.

**G. 557.** An aquarium with an open top has a shape of a square-based right prism. Its base and face walls are made of 1 cm thick glass sheet. The inner height of the aquarium is 20 cm, and each of its inner base edge is 30 cm long. The aquarium is filled with water. The water flows into the aquarium from a tap at a rate of 5 cm$^3$ per second. *a*) How long does it take to fill the half of the aquarium with water? *b*) What is the weight of the halfway filled aquarium? (The density of water is 1000 kg/m$^3$, and the density of glass is 2500 kg/m$^3$.) **G. 558.** Six forces are exerted on the same body at the same time: $F_1=1$ N, $F_2=2$ N, $F_3=3$ N, $F_4=4$ N, $F_5=5$ N and $F_6=6$ N. The forces are all in the same plane and the angles between two adjacent forces are 60° (i.e. the consecutive rotations of the forces are always 60° in the same direction). *a*) What is the vector sum of the six forces? *b*) How should the magnitude or maybe as well the direction of force $F_2$ be changed in order for the object to be in equilibrium? **G. 559.** We have two sets of Christmas lights, both rated at 50 W and 230 V. In one of the sets there are 50 bulbs, whilst in the other there are 100 bulbs connected in series. *a*) In which set is the current greater? *b*) In which set is the resistance of a bulb greater? *c*) Will the power of the set containing 100 bulbs increase or decrease if 10 of its bulbs are changed to ten bulbs from the other set? (Assume that none of the bulbs blow out.) **G. 560.** One end of an initially 2 m long horizontal elastic band is tied to a wall, and a snail is moving along it at a speed of 1 m/h. The snail starts from the wall, and the band is stretched by 1 m from the end of the band at the end of each hour. How much time elapses until the snail reaches the end of the band?

**P. 5089.** The frictionless trajectory shown in the figure consists of two circular arcs. A tiny object starts to slide from point $A$ along the trajectory with a very small initial speed. How long does it take for the tiny object to reach the right end of the curved path (point $B$)? **P. 5090.** There is a cube-shaped box of mass $m$ on a horizontal floor. A thin, uniform rod of also mass $m$ is leant against one of the faces of the cube touching it at its centre. Initially both of the objects are fixed. The angle between the ground and the rod is $\alpha=45°$. What is the initial acceleration of the box at which it starts to move when the objects are released? (Friction is negligible everywhere.) **P. 5091.** At STP the density