MATHEMATICAL AND PHYSICAL JOURNAL FOR SECONDARY SCHOOLS
(Volume 68. No. 6. September 2018)

Problems in Mathematics

New exercises for practice – competition K (see page 352): **K. 589.** A company regularly hires the same man to mow a certain grassy area. In addition to the labour cost, the man charges a call-out fee of 5000 forints (HUF, Hungarian currency). If he mows the grass three times a month, the labour cost is 1.8 times as much per occasion as it is in the case when he mows the grass four times a month, since in the former case the grass will grow higher which requires more work. On the whole, it is more economical for the company to have the grass mowed four times a month. What is the minimum possible labour cost per mowing in the case of 4 occasions a month, given that it is an integer multiple of 100 forints? **K. 590.** Andrew, Bill and Charlie participated in a running race. At the end, Bill and Carlie, respectively, were 15 metres and 35 metres behind when Andrew finished. When Bill finished, Charlie was still 22 metres behind him. What was the distance covered by the runners, given that each boy was running at a uniform speed? **K. 591.** A group of children participated in a bus excursion. In the luggage compartment of the bus, there is room for the luggage of one third of the children. The bus has 52 seats. The teachers are sitting on two seats. On each of the remaining seats, there is either a child sitting, or there is the luggage of a child. How many children are there on the trip if all seats are used? **K. 592.** Ann, Belle and Charlotte are doing a job together. The three of them finish it 6 hours sooner than Ann would finish it alone, 1 hour sooner than Belle would finish it alone, and they take half as much time together as Charlotte would take alone. If Ann and Belle work together without Charlotte, they finish the job in 80 minutes. How long would it take each girl Ann and Belle to do the job alone? (Express your answer in minutes.) **K. 593.** A disobedient class were trying to boycott the PE lesson. Therefore, when given the task of throwing a ball along the running track as far as they can, they threw the balls towards the fence instead, in the hope of getting the balls across the fence and halting the lesson while the balls are collected. 1/6 of the students threw 5 balls each, half the class threw 4 balls per student, one student threw 6 balls and all the rest of them threw 2 balls each. They managed to throw 75% of the balls over the fence. Since it took a lot of of time to get all these 66 balls back, the students were punished. They were made to run 3 laps each on the 200-metre running track. How many kilometres did they run altogether?

New exercises for practice – competition C (see page 353): **Exercises up to grade 10: C. 1490.** What is the remainder if the number $N=863\underbrace{99\ldots9}_{2018\text{ times}}$ (with 2018 digits of 9 at the end) is divided by 32? **C. 1491.** The length of side AD of a rectangle $ABCD$ is 1 cm. The angle bisector of $\angle BAD$ and the perpendicular bisector of diagonal AC intersect on the side CD. Find the exact length of side CD. **Exercises for everyone: C. 1492.** The solid in the *figure* consists of 15 unit cubes. In how many different ways is it possible to get from vertex A to vertex B along the grid lines if it is only allowed to move in the three directions indicated? **C. 1493.** A triangle of unit area has sides a, b, c, such that $a\geqslant b\geqslant c$. Show that $b\geqslant\sqrt{2}$. **C. 1494.** Prove that if p and q are twin primes greater than 3, then their arithmetic mean is divisible by 6 and the number greater by 1 than their product is divisible by 36. **Exercises upwards of grade 11: C. 1495.** Consider the following sequence of equalities: (1) $1+2=3$, (2) $4+5+6=7+8$, (3) $9+10+11+12=13+14+15$. By observing the rule, state the kth row, and prove your statement. (Proposed by *Á. Kertész,* Miami Beach) **C. 1496.** Disks of radii 1, 2 and

3 cm are drawn about the vertices of a triangle. The three disks pairwise touch each other from outside. What is the area in the triangle that is not covered by the disks?

New exercises – competition B (see page 354): **B. 4966.** Find the nineteenth positive integer in which the sum of the digits is 2018. (*3 points*) **B. 4967.** Let P be an interior point of $\triangle ABC$, and the midpoint of sides AB, BC and CA are C_1, A_1 and B_1, respectively. Through the points A_1, B_1 and C_1, draw parallels to the lines AP, BP and CP, respectively. Show that these three lines are concurrent. (*3 points*) (Proposed by *J. Kozma,* Szeged) **B. 4968.** Solve the following simultaneous equations on the set of positive real numbers:

$$\frac{1}{1+a+ab+abc}+\frac{1}{1+b+bc+bcd}+\frac{1}{1+c+cd+cda}+\frac{1}{1+d+da+dab}=1,$$

$$a+b+c+d=4.$$

(*4 points*) **B. 4969.** The sides of a rectangle T are $a \leqslant b$. Given that the union of two appropriate disks of radius r covers T but this is impossible with two disks of radii smaller than r, determine the length r. (*4 points*) **B. 4970.** A line e in the plane separates two given points A and B of the plane. On the line e, select points P and Q such that $\angle PAQ = 90^\circ$. Show that there exists a point different from B that lies on all circles drawn through B, P and Q, independently of the choice of P and Q. (*5 points*) (Proposed by *the class 11C* of Fazekas Primary and Secondary School, Budapest) **B. 4971.** For which primes p is there a positive integer a such that $1+a+a^2+\dots+a^{p-1}$ is divisible by p^2? (*5 points*) **B. 4972.** Let P be an interior point of an acute-angled triangle ABC. Let D, E and F be the orthogonal projections of P onto the sides, as shown in the *figure.* Outside the triangle, a square is drawn over each of the six line segments formed on the sides. The squares are then alternatively coloured by two colours according to the figure. Consider the two triangles formed by the lines of the "outer" sides of the squares with the same colour. Show that these two triangles are congruent. (*6 points*) **B. 4973.** Let $a_1, a_2, \dots, a_{2018}$ denote non-negative real numbers that add up to 1. Find the largest possible value of the sum $S=\sum_{i\neq j,\ i|j} a_i a_j$ (*6 points*) (Based on an *Argentinean problem*)

New problems – competition A (see page 356): **A. 728.** Floyd the flea makes jumps on the positive integers. On the first day he can jump to any positive integer. From then on, every day he jumps to another number that is not more than twice his previous day's place. *a*) Show that Floyd can make infinitely many jumps in such a way that he never arrives at any number with the same sum of decimal digits as at a previous place. *b*) Can the flea jump this way if we consider the sum of binary digits instead of decimal digits? (*Dürer competition,* 2015) **A. 729.** In a cyclic quadrilateral $ABCD$, the diagonals meet at point E, the midpoint of side AB is F, and the feet of perpendiculars from E to the lines DA, AB, and BC are P, Q, and R, respectively. Prove that the points P, Q, R, and F are concyclic. (Proposed by *M. Weisz,* Szeged) **A. 730.** Let F_n be the nth Fibonacci number ($F_1=F_2=1$, $F_{n+1}=F_n+F_{n-1}$). Construct infinitely many positive integers n such that n divides F_{F_n} but n does not divide F_n.

Problems in Physics

(see page 378)

M. 379. Measure the height from which a retractable ballpoint click pen should be dropped from a vertical position in order that it is clicked on. When the tip of the pen is already out, from what height should it be dropped vertically in order that it is clicked off?

G. 641. The mass of each pulley in the pulley system shown in the *figure* is 1 kg. What is the magnitude of the force F with which the monkey in the cage can be kept in equilibrium? The total mass of the monkey and the cage is 9 kg. **G. 642.** A wheel of radius R is rolling without slipping along the inner circumference of a circle of radius $2R$. What is the path of a point on the rim of the small wheel? **G. 643.** A bowl of water is placed on one of the plates of a beam scale, which is balanced. One of our fingers is immersed into the water, such that the bowl is not touched. Will the scale remain balanced or not, if *a*) not a drop of water flows out of the bowl; *b*) the bowl was full to the brim, and the water flows off the plate of the scale? **G. 644.** Four alike bulbs are connected as shown in the *figure* to a battery. Which one is the brightest? List them in the order of their luminous power.

P. 5045. An object falls freely on the Moon. The total height from which the object was dropped is n times greater than the distance covered by the object in the last second of its fall. Determine the height from which it was dropped and the time of the fall. **P. 5046.** A board of mass M was fixed symmetrically as shown in the *figure.* The masses of the fixed pulleys and the ropes, as well as friction in the shaft are negligible. (The objects of mass m are not glued to the board of mass M.) At what m/M ratio will the system be in equilibrium? **P. 5047.** The cart of mass M and the flat block of mass m on the cart are moving at a speed of v towards a compression spring fixed to a wall, and having a spring constant of D. The coefficient of friction between the block and the cart is μ. *a*) Will the block slip or not when the collision occurs? *b*) How long does the collision last? *Data:* $M = 0.2$ kg, $m = 0.1$ kg, $v = 1$ m/s, $D = 4.4$ N/m, $\mu = 0.4$. **P. 5048.** Estimate the pressure difference in the tyre of a car between an „inner” point next to the rim of the wheel, and an „outer” point next to the tread area. The excess pressure of the tyre when the car is at rest is 2 bars, and the maximum allowed speed on the motorway is 130 km/h. **P. 5049.** To what minimum speed must a rocket be accelerated, in order that it reaches the Moon from the Earth? Compare this speed to the escape speed on Earth. (For the sake of simplicity, do not consider the motion of the Earth and the Moon.) **P. 5050.** There is some water and some water vapour above it in a closed container. What happens to the water if the water vapour is sucked out of the container, whilst the temperature of the water is kept constant, if this temperature is *a*) 100.00 °C; *b*) 20.00 °C? **P. 5051.** A sample of helium is taken through the processes shown in the *figure.* How much thermal energy is absorbed during the process? **P. 5052.** There is a small grain under a 1.5 cm wide glass sheet. Where is the visible image of the grain, if the light rays are perpendicular to the surface of the sheet, and the refractive index of the glass is $n = 1.5$? **P. 5053.** Determine the total luminosity of a star whose surface temperature is 7500 K, and whose diameter is 2.5 times bigger than that of the Sun. The surface temperature of the Sun is 5800 K, and express the luminosity of the star in terms of the luminosity of the Sun (which is considered to be 1). **P. 5054.** A nucleus of rest mass M, which is initially at rest can absorb a gamma quantum of energy hf. Determine the excitation energy of this nucleus in the process. (So by what amount does its rest energy increase in the process?) **P. 5055.** One end of a horizontally stretched rubber thread is moved periodically in the vertical direction, so transverse waves are travelling along the thread. A film is recorded about the motion of a small part of the thread, and three consecutive frames of this film are shown in the *figure.* In which direction does the energy propagate in the rope, from the left to the right or from the right to the left?