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Problems in Mathematics

New exercises for practice – competition C (see page 286): **Exercises up to grade 10: C. 1483.** What is the smallest value of the expression $6|x-1|+5|x-2|+4|x-3|+3|x+4|+2|x-5|$? **C. 1484.** The diagonals of a convex quadrilateral $ABCD$ are not perpendicular. The feet of the perpendiculars dropped from vertices A, B, C, D onto the sections AC and BD are A_1, B_1, C_1, D_1, respectively. (They are different from vertices.) Prove that these points form a quadrilateral similar to the original one. **Exercises for everyone: C. 1485.** Let $x=1^2+3^2+5^2+\dots+2017^2$ and $y=2^2+4^2+6^2+\dots+2018^2$. Evaluate the fraction $\frac{y-x}{y+x-(1\cdot 2+2\cdot 3+3\cdot 4+\dots+2017\cdot 2018)}$. **C. 1486.** A regular triangle ABC and a circle k are both centred at point O, and have an equal area of $\sqrt{\frac{\pi}{\sqrt{27}}}$. Let the extensions of line segments AO, BO, CO intersect circle k at points A', B', C', respectively. Find the exact value of the area of the hexagon $AC'BA'CB'$. **C. 1487.** Nine actors take part of acting exercises each involving three characters. With what minimum number of exercises is it possible to make sure that every pair of actors play together at least once? **Exercises upwards of grade 11: C. 1488.** Given that any three line segments out of a set of five can be used as sides to construct a (non-degenerate) triangle, prove that at least one of the triangles obtained in this way is acute-angled. **C. 1489.** In the lower left corner of a chessboard there is a black bishop, and in the lower right corner there is a white bishop. Each bishop moves up the board in steps of one unit, remaining on fields of its own colour. In each step, it may move to the left or to the right, at random, until it reaches the top row. What is the probability that the black bishop ends up to the right of the white bishop?

New exercises – competition B (see page 287): **B. 4957.** A set with positive integer elements is said to be *jolly good* if it does not contain a pair of numbers whose difference is 2. How many *jolly good* subsets does the set $\{1,2,3,\dots,10\}$ have? (*3 points*) (Proposed by *S. Róka,* Nyíregyháza) **B. 4958.** The sides of a triangle are a, b, c, the radius of the inscribed circle is r, and the radius of the circumscribed circle is R. Prove that if $a+b+c=\frac{4}{rR}$, $\sqrt{ab}+\sqrt{bc}+\sqrt{ca}=6$ then $R=2r$. (*4 points*) (*Romanian competition problem*) **B. 4959.** Barnaby has n marbles in his pocket. When he performs a somersault, each marble has a probability of $0<p<1$ to fall out of his pocket, independently of one another. If at least one marble falls out during a somersault then Barnaby will stop performing somersaults. Otherwise he will continue. Given that the probability of having an even number of marbles in his pocket when he stops doing somersaults is 50%, what may be the value of n? (*4 points*) **B. 4960.** Let P be an interior point of triangle ABC, and let A^*, B^* és C^* be arbitrary points of the line segments AP, BP and CP, respectively. Through point A^*, draw parallels to BP and CP, which intersect sides AB and AC at A_1 and A_2, respectively, as shown in the *figure.* Similarly, the parallels drawn through point B^* to CP and AP intersect sides BC and AB at B_1 and B_2, and finally, the parallels drawn through point C^* to AP and BP intersect sides AC and BC at C_1 and C_2, respectively. Show that $AC_1\cdot BA_1\cdot CB_1=AB_2\cdot BC_2\cdot CA_2$. (*3 points*) (Proposed by *J. Kozma,* Szeged) **B. 4961.** The intersection of three unit circles is bounded by the arcs $\widehat{AB}$, $\widehat{AC}$ and $\widehat{BC}$. The perimeter of the intersection is K. Calculate the perimeter of the intersection of the unit circles centred at A, B and C. (*4 points*) **B. 4962.** Let n be a positive integer. Solve the following simultaneous equations on the set of real numbers: $a_1^2+a_1-1=a_2$, $a_2^2+a_2-1=a_3$, $\dots$, $a_n^2+a_n-1=a_1$. (*5 points*) **B. 4963.** Let r_a be the radius of the largest escribed circle of a triangle, and let R denote the radius of the circumscribed circle. Prove that $r_a \geqslant \frac{3}{2}R$. (*5 points*) (A problem from *Paul Erdős* (1913–1996)) **B. 4964.** Is it true that if the functions $f, g : \mathbb{R} \to [0,1]$ are periodic and the function $f+g$ is also periodic then they have a period in common? (*6 points*) **B. 4965.** For a vector $\mathbf{x} \neq \mathbf{0}$, let $\mathbf{e}_{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$. A given plane $\mathcal{S}$ is parallel, but not identical to the plane of a given (non-degenerate) triangle ABC. Show that there exists a unique $P \in \mathcal{S}$, such that the vector $\mathbf{e}_{\overrightarrow{PA}} + \mathbf{e}_{\overrightarrow{PB}} + \mathbf{e}_{\overrightarrow{PC}}$ is perpendicular to $\mathcal{S}$. (*6 points*)

New problems – competition A (see page 289): **A. 725.** Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following equation for all $x, y \in \mathbb{R}^+$: $f\big(xy + f(y)^2\big) = f(x)f(y) + yf(y)$. (Proposed by: *Ashwin Sah,* Cambridge, Massachusetts, USA) **A. 726.** In triangle ABC with incenter I, line AI intersects the circumcircle of ABC at $S \neq A$. Let the reflection of I with respect to BC be J, and suppose that line SJ intersects the circumcircle of ABC for the second time at point $P \neq S$. Show that $AI = PI$. (Proposed by: *József Mészáros,* Galanta, Slovakia) **A. 727.** For any finite sequence $(x_1, \ldots, x_n)$, denote by $N(x_1, \ldots, x_n)$ the number of ordered index pairs (i, j) for which $1 \leqslant i < j \leqslant n$ and $x_i = x_j$. Let p be an odd prime, $1 \leqslant n < p$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be arbitrary residue classes modulo p. Prove that there exists a permutation π of the indices $1, 2, \ldots, n$ for which $N\big(a_1 + b_{\pi(1)}, a_2 + b_{\pi(2)}, \ldots, a_n + b_{\pi(n)}\big) \leqslant \min\big(N(a_1, a_2, \ldots, a_n), N(b_1, b_2, \ldots, b_n)\big)$.

Problems in Physics

(see page 314)

M. 378. Measure by what percent does a mosquito net (or any similar material) decrease the transparency of a window.

G. 637. Two balls are started at the same initial speed, each rolls along a horizontal plane first. During the motions both balls roll down along a slope, and then they both roll up to the initial level of their motion, and then they got to the end of the paths. The lengths of both paths are the same, the depths of the paths are also the same. Friction is negligible in both cases. Which ball reaches the end of the path first? **G. 638.** A 2-ton vehicle without operating its engine would move down a slope of 5% elevation at a constant speed of 36 km/h. What would the useful power of its engine be if it goes up along the same slope at the same speed? **G. 639.** It was observed that the flames of a burning candle has a spherical shape in a spaceship revolving around the Earth. Explain this observation. **G. 640.** If the Earth was a uniform density sphere of radius R, which of the *graphs* shown below would be the correct sketch of the gravitational force as a function of the distance measured from the centre of the Earth?

P. 5034. How long did the object, projected horizontally at an initial speed of v_0, fall while it reached a position which was at a distance of s from the position of the projection? (Neglect air resistance.) *Data:* $v_0 = 5$ m/s, $s = 20$ m. **P. 5035.** In winter a favourite type of food for titmice is a fat ball consisting of fat and different seeds. These balls are suspended by means of a piece of thread and hung to a branch of a tree. Even two tits can feed themselves from the same ball at the same time. Once there were two tits on the same ball of mass 90 g, when suddenly they got frightened and flew off at the same moment, with the same initial speed, in perpendicular directions, such that both tits initial velocity made an angle of 35° degree with the horizontal. The fat ball began to swing with a period of 1.4 s, the angular displacement of the thread (with respect to the vertical) was 10°. The mass of each titmouse is 18 g. What was the initial speed