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Problems in Mathematics

New exercises for practice – competition C (see page 224): **Exercises up to grade 10: C. 1476.** Prove that the inequality $\frac{(y-6)^2}{3xy}+x\cdot\frac{y+3}{y}\geqslant 4+x-\frac{4}{x}-\frac{xy}{12}$ holds for all positive x and y. **C. 1477.** Prove that if there is a point E on base AD of a trapezium $ABCD$ such that the perimeters of triangles ABE, BCE and CDE are equal then $BC=\frac{1}{2}AD$. **Exercises for everyone: C. 1478.** Given that a six-digit number is divisible by 37, its digits are all different, and 0 does not occur among them, show that at least six more numbers divisible by 37 can be obtained by changing the order of the digits. **C. 1479.** In a triangle ABC, T is an interior point of side AC such that $TA=BC$, and P is an interior point of side AB such that the triangles CBP and PAT are congruent. Q is an interior point of side BC such that TQ is not parallel to AB and triangle BPQ is similar to triangle TCQ. Prove that $PT=QT$. **C. 1480.** Solve the equation $\frac{x^3-7x+6}{x-2}=\frac{2x+14}{x+2}$ on the set of integers. **Exercises upwards of grade 11: C. 1481.** The vertices of a regular octagon inscribed in a circle of radius 2 are connected in three different ways, as shown in the *figure:* each vertex with the adjacent vertices, each vertex with the second adjacent vertices, and finally, each vertex with the third adjacent vertices. Prove that the product of the radii of the three inscribed circles is 2. **C. 1482.** Prove that $\left|2\sin x+\sin(2x)\right|<\frac{3+2\sqrt{2}}{2}$.

New exercises – competition B (see page 226): **B. 4948.** The positive integer n is said to be *chunky* if it has a prime factor greater than $\sqrt{n}$. For example, 2017 (a prime number), $2018=2\cdot 1009$ and $2022=2\cdot 3\cdot 337$ are chunky, while $2023=7\cdot 17^2$ is not. How many chunky numbers are there which only have prime factors less than 30? (*3 points*) (Proposed by *S. Róka,* Nyíregyháza) **B. 4949.** The feet of the altitudes drawn from vertices B and C of an acute-angled triangle ABC are D and E, respectively. Let P be an interior point of AD, and let Q be an interior point of AE such that $EDPQ$ is a cyclic quadrilateral. Show that the line segments BP and CQ intersect on the median drawn from A. (*3 points*) **B. 4950.** Let F_n denote the nth Fibonacci number ($F_1=F_2=1$, $F_{n+2}=F_{n+1}+F_n$), and define the sequence $a_0,a_1,a_2,\dots$ with the following recurrence relation: let $a_0=2018$, and for all $k\geqslant 0$ let $a_{k+1}=a_k+F_n$, where F_n is the largest Fibonacci number less than a_k. Will there be any Fibonacci number in the sequence (a_k)? (*4 points*) **B. 4951.** The elements of a set V are n-dimensional vectors (ordered n-tuples of numbers) of which each coordinate is -1, 0 or 1. No three different vectors of V add up to the zero vector. Show that $|V|\leqslant 2\cdot 3^{n-1}$. (*4 points*) **B. 4952.** Is it possible to dissect a cube with a finite number of straight cuts so that the pieces can be put together to form two smaller congruent cubes? (*5 points*) (Proposed by *Z. Gyenes,* Budapest) **B. 4953.** Prove that $\ln n+\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\ldots+\sqrt{\frac{n-1}{n}}<\sqrt{2}+\sqrt{\frac{3}{2}}+\sqrt{\frac{4}{3}}+\ldots+\sqrt{\frac{n}{n-1}}$ for all integers $n>1$. (*5 points*) (Proposed by *G. Holló,* Budapest) **B. 4954.** Line ℓ passes through vertex A of a triangle ABC, and it is parallel to BC. Let ℓ intersect the interior angle bisectors of angles ABC and ACB at K and L, respectively. The inscribed circle touches BC at point D. Show that the circumscribed circle intersects the Thales circle of line segment KL at two points, and these two points are collinear with D. (*6 points*) **B. 4955.** Let n be a positive integer. What is the largest possible number of ordered triples of non-negative integers $(x_1,y_1,z_1),(x_2,y_2,z_2),\dots$ such that the following conditions hold: (1) For all i, $x_i+y_i+z_i=n$. (2) The numbers $x_1,x_2,\dots$ are all different, the numbers

$y_1, y_2, \ldots$ are all different, and the numbers $z_1, z_2, \ldots$ are also all different. Give an example for such a sequence of maximum length with the required property. (*6 points*) (Proposed by *P. Erben,* Budapest) **B. 4956.** A transformation of central similitude is applied to a tetrahedron $ABCD$ with each vertex as centre. The four diminished tetrahedra obtained are $AA_bA_cA_d$, $B_aBB_cB_d$, $C_aC_bCC_d$ and $D_aD_bD_cD$. Given that these small tetrahedra are pairwise disjoint, prove that the volumes of tetrahedra $A_bB_cC_dD_a$, $A_bB_dD_cC_a$, $A_cC_bB_dD_a$, $A_cC_dD_bB_a$, $A_dD_bB_cC_a$ and $A_dD_cC_bB_a$ are equal. (*6 points*) (Proposed by *Sz. Kocsis,* Budapest)

New problems – competition A (see page 227): **A. 722.** The Hawking Space Agency operates $n-1$ space flights between the n habitable planets of the Local Galaxy Cluster. Each flight has a fixed price which is the same in both directions, and we know that using these flights, we can travel from any habitable planet to any habitable planet. In the headquarters of the Agency, there is a clearly visible board on a wall, with a portrait, containing all the pairs of different habitable planets with the total price of the cheapest possible sequence of flights connecting them. Suppose that these prices are precisely $1, 2, \ldots, \binom{n}{2}$ monetary units in some order. Prove that n or $n-2$ is a square number. **A. 723.** Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that the limit $g(x)=\lim\limits_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ exists for all real x. Prove that $g(x)$ is constant if and only if $f(x)$ is a polynomial function whose degree is at most 2. **A. 724.** A sphere $\mathcal{G}$ lies within tetrahedron $ABCD$, touching faces ABD, ACD, and BCD, but having no point in common with plane ABC. Let E be the point in the interior of the tetrahedron for which $\mathcal{G}$ touches planes ABE, ACE, and BCE as well. Suppose the line DE meets face ABC at F, and let L be the point of $\mathcal{G}$ nearest to plane ABC. Show that segment FL passes through the centre of the inscribed sphere of tetrahedron $ABCE$.

Problems in Physics

(see page 250)

M. 377. Create a short circuit across a solar cell by connecting an ammeter across it. Investigate how the current depends on the angle of incidence of the „direct sun rays". Do not forget to set a proper measurement range for the ammeter.

G. 633. Is it possible that after a free kick is awarded to a team on a soccer pitch, the ball bounces from the crossbar of the goal behind the goal line and from the ground it bounces outward towards the pitch? **G. 634.** The inner ring of the ball-bearing shown in the *figure* is at rest, the centres of the balls are undergoing circular motion at a speed of 0.2 m/s. What is the number of revolution of the outer ring, if $r=3$ cm, $R=4$ cm? **G. 635.** There is some water at a temperature of 0 °C in a bowl. Some part of it is taken out and frozen to a piece of ice at a temperature of 0 °C, then it is put back to the bowl, in which it floats on the top of the remaining water in it. *a*) Will the top of the ice be at a greater height than the original water level? *b*) Which one has the greater gravitational potential energy: the water ice system, or the original water in the bowl? **G. 636.** How far may the city Miskolc be from the road sign in which the following warning is written: "You cannot even save 8 minutes till Miskolc. Is it worth it?" (The speed limit for the highways in Hungary is 130 km/h.)

P. 5023. The speed of an object sliding down a slope of angle of elevation 25°, is one-quarter of the final speed that the object could have reached if there was no friction. What is the coefficient of kinetic friction? **P. 5024.** An object of mass m is suspended by a rubber thread of unstretched length l, and of force (spring) constant k. Then the