A HENGERESEN ORTOTRÓP NÉGYZET KERESZTMETSZETŰ RÚD SAINT-VENANT FÉLE CSAVARÁSI FELADATÁNAK KÖZELÍTŐ MEGOLDÁSA

APPROXIMATE SOLUTION OF SAINT-VENANT TORSION OF CYLINDRICALLY ORTHOTROPIC BAR WITH SQUARE CROSS SECTION

Ecsedi István, Professor Emeritus, Miskolci Egyetem, Műszaki Mechanikai Intézet;
Baksa Attila, egyetemi docens, PhD, Miskolci Egyetem, Műszaki Mechanikai Intézet;
Lengyel Ákos József, adjunktus, PhD, Miskolci Egyetem, Műszaki Mechanikai Intézet

ABSTRACT. This paper deals with the Saint-Venant torsion of elastic cylindrical orthotropic solid square cross section. The origin of the cylindrical orthotropy coincides with the center of square cross section. By the use of two minimum theorems of elasticity upper and lower bounds are derived for the torsional rigidity of the considered cross section. Illustrative example shows that the one term approximation leads to relative closed bounds for the torsional rigidity.

ÖSSZEFOGLALÁS: A dolgozat négyzetkeresztmetszetű hengeresen anizotrop rúd Saint-Venant csavarási feladatával foglalkozik. A rugalmasságtan két minimum tételének felhasználásával alsó és felső korlátokat bizonyít a keresztmetszet csavarási merevségére. Példa szemlélteti a levezetett összefüggések alkalmazását.

Keywords: Saint-Venant torsion, cylindrically orthotropic, torsional rigidity, upper and lower bounds, shearing stresses.

1 INTRODUCTION

While the uniform torsion of homogeneous Cartesian anisotropic linearly elastic bars has been well documented it is the subject of several studies from both theoretical and numerical viewpoints [1-6] until then relative few articles and books deal with the task of uniform torsion problem of cylindrically anisotropic bars [2,3,4,7-11]. The object of this paper is the Saint-Venant torsion of homogenous cylindrical orthotropic solid square cross-section. The bar with square cross section is an important structural component, the investigation of its deformation under the torsional load is the subject of several books of linear elasticity [1-4].

Figure 1 shows the bar with square cross section which is subjected to torsional load. The material of the bar is elastic, homogenous and cylindrical orthotropic with shear moduli $G_{rz}, G_{\varphi z}$ [2,3,4].

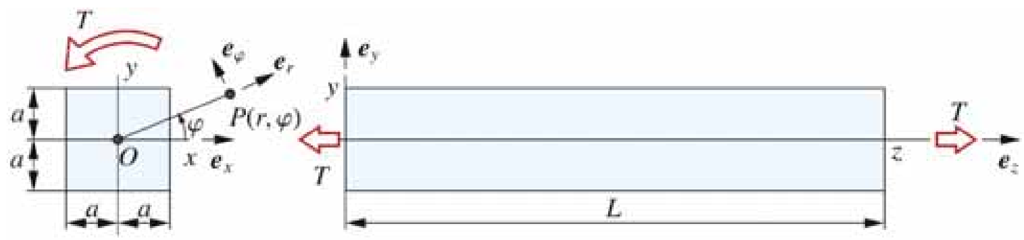

Figure 1 Cylindrical anisotropic bar with torsional load.

The Prandtl stress function formulation in the cylindrical coordinate system $Or\varphi z$ leads to the following Dirichlet type boundary-value problem (Figure 1) for the uniform torsion of the cylindrical orthotropic cross section

$$\frac{1}{G_{\varphi z}}\frac{\partial^2 U}{\partial r^2}+\frac{1}{rG_{\varphi z}}\frac{\partial U}{\partial r}+\frac{1}{r^2 G_{rz}}\frac{\partial^2 U}{\partial \varphi^2}=-2 \qquad (x,y)\in A, \tag{1}$$

$$U=0 \qquad (x,y)\in \partial A. \tag{2}$$

In equations (1) and (2) $r,\ \varphi$ are cylindrical coordinates, $U=U(r,\varphi)$ is the Prandtl stress function of the considered cross section. The cross section of the bar is A and the boundary curve of A is denoted by ∂A, that is, in present problem (Figure 1)

$$A=\{(x,y)|-a<x<a,-a<y\le a\}, \tag{3}$$

$$\partial A=\left\{\begin{matrix}(x,y)|\,(a^2-x^2)(a^2-y^2)=0,\\ 0<|x|\le a, 0\le y\le a\end{matrix}\right\}. \tag{4}$$

The definition of cross-sectional Descartes coordinates x, y and z are give in the Figure 1. The torsional rigidity of the cross section in the framework of Saint-Venant theory is defined as

$$S = \frac{T}{\vartheta}, \qquad (5)$$

where T is the applied torque, ϑ is the rate of twist with respect to the axial coordinate z. The shearing stresses τ_{rz} and $\tau_{\varphi z}$ obtained from equations (6) and (7)

$$\tau_{rz} = \frac{\vartheta}{r}\frac{\partial U}{\partial \varphi} = \frac{T}{S}\frac{1}{r}\frac{\partial U}{\partial \varphi} \quad (r,\varphi) \in A \cup \partial A, \qquad (6)$$

$$\tau_{\varphi z} = -\vartheta\frac{\partial U}{\partial r} = -\frac{T}{S}\frac{\partial U}{\partial r} \quad (r,\varphi) \in A \cup \partial A. \qquad (7)$$

It is known that the solution of the Saint-Venant torsion of orthotropic bar can be obtained as the solution of the following variational problem according to the principle of minimum of complementary energy.

$$\Pi_c[U(r,\varphi)] = \min_{\tilde{U}(r,\varphi)} \Pi_c[\tilde{U}(r,\varphi)] \qquad (8)$$

where $\tilde{U} = \tilde{U}(r,\varphi)$ denotes the statically admissible stress function which satisfies a homogeneous boundary condition on the boundary curve ∂A

$$U(r,\varphi) = 0, \quad (r,\varphi) \in \partial A. \qquad (9)$$

Explicit form of $\Pi_c\left[\tilde{U}(r,\varphi)\right]$ is

$$\Pi_c\left[\tilde{U}(r,\varphi)\right] = 4\int_A \tilde{U}\,(x,\varphi)\mathrm{d}A - \int_A \left[\frac{1}{G_\varphi}\left(\frac{\partial \tilde{U}}{\partial r}\right)^2 + \frac{1}{r^2 G_r}\left(\frac{\partial \tilde{U}}{\partial \varphi}\right)^2\right] dA. \qquad (10)$$

In equation (10) $\tilde{U} = \tilde{U}(r,\varphi)$ is a continuous function of its arguments r and φ in $\overline{A} = A \cup \partial A$ and it has continuous second order partial derivatives with respect to r and φ in A [12,13,14]. In paper [15], Ecsedi and Lengyel proved the following lower bound for the torsional rigidity of cylindrically orthotropic bar, when

$$A_{44} = G_{\varphi z}, A_{55} = G_{rz}, A_{45} = A_{54} = 0 \qquad (11)$$

$$S \le S_U = \int_A G_{\varphi z}\mathrm{d}A - \frac{\left(\int_A G_{\varphi z}\frac{\partial \tilde{\omega}}{\partial \varphi}\mathrm{d}A\right)^2}{\int_A \left[G_{rz}\left(\frac{\partial \tilde{\omega}}{\partial r}\right)^2 + G_{\varphi z}\frac{1}{r^2}\left(\frac{\partial \tilde{\omega}}{\partial \varphi}\right)^2\right]\mathrm{d}A} \qquad (12)$$

In formula (11) $\tilde{\omega} = \tilde{\omega}(r,\varphi)$ is an arbitrary function whose second order partial derivatives with respect to r and φ are continuous functions in $A \cup \partial A$. The name of $\tilde{\omega} = \tilde{\omega}(r,\varphi)$ is kinematically admissible torsion function [12,13,14,15]. The proof of upper bound formula (12) is based on the principle of the minimum of potential energy [15].

2 APPROXIMATE SOLUTION FOR THE PRANDTL STRESS FUNCTION

The solution of the variational problem (8) is searched as

$$\tilde{U}(r,\varphi) = c(a^2 - r^2\cos^2\varphi)(a^2 - r^2\sin^2\varphi) = c(a^4 - a^2r^2 + r^4\cos^2\varphi\sin^2\varphi) \quad (r,\varphi) \in A \cup \partial A. \qquad (13)$$

It is evident that

$$\tilde{U}(r,\varphi) = 0 \quad (r,\varphi) \in \partial A \qquad (14)$$

independently of the value of unknown constant c. The stacionary condition of the compementary energy functional (10) under the boundary condition (14) yields the following result

$$\Pi_c(c) = c\frac{64}{9}a^6 - c^2\left(-\frac{64}{9}G_{\varphi z}a^8 + \frac{64}{5}G_{rz}a^8 + 4\pi G_{\varphi z}a^8 - 4\pi G_{rz}a^8\right). \qquad (15)$$

From the principle of minimum of complementary energy it follows that the best approximate analytical solution satisfies

$$\frac{\mathrm{d}\Pi}{\mathrm{d}e} = 0 \qquad (16)$$

the stationary condition, which gives

$$c = \frac{40}{ka^2} \qquad (17)$$

where

$$k = (45\pi - 80)G_{\varphi z} + (144 - 45\pi)G_{rz} \qquad (18)$$

Final form of $\tilde{U} = \tilde{U}(r,\varphi)$ can be represented as

$$\tilde{U}(r,\varphi) = \frac{40}{ka^2}(a^4 - a^2r^2 + r^4\cos^2\varphi\sin^2\varphi) \qquad (19)$$

Substitution the expression of $\tilde{U} = \tilde{U}(r,\varphi)$ into the equation (6) and (7) gives the expressions of shearing stresses

The resultant of shearing stress $\tilde{\tau}_z = \tilde{\tau}_z(r,\varphi)$ is as follows

$$\tilde{\tau}_z(r,\varphi) = \sqrt{\tilde{\tau}_{rz}^2 + \tilde{\tau}_{\varphi z}^2} = \frac{80\vartheta r}{ka^2}(4a^2r^2\cos^4\varphi - r^4\cos^4\varphi - 4a^2r^2\cos^2\varphi + a^4 + 4r^3\cos^4\varphi)^{\frac{1}{2}} \tag{22}$$

3 LOWER AND UPPER BOUNDS FOR THE TORSIONAL RIGIDITY

It is known that

$$\Pi_c\left[\widetilde{U}(r,\varphi)\right] = S_L \leq S \tag{23}$$

the lower bound relation (23) is valid with arbitrary statically admissible $\widetilde{U} = \widetilde{U}(r,\varphi)$ stress function and equality in bounding formula is reached only if $\widetilde{U}(r,\varphi) = U(r,\varphi)$ $(r,\varphi) \in A \cup \partial A$ [12, 13, 14] In the present problem by a single computation, the following lower bound can be derived for S

$$S \geq S_L = \frac{1280}{9}\frac{a^4}{k} \tag{24}$$

Substitution

$$\widetilde{\omega}(r,\varphi) = \frac{r^4}{4}\sin(4\varphi) \quad (r,\varphi) \in A \cup \partial A \tag{25}$$

into the expression of the upper bound formula (12) gives

$$S \leq S_U = \frac{8G_{\varphi z}a^4}{45k_U}[(1680 - 525\pi)G_{rz} + (525\pi - 1604)G_{\varphi z}] \tag{26}$$

where

$$k_U = (112 - 35\pi)G_{rz} + (35\pi - 106)G_{\varphi z} \tag{27}$$

For isotropic bar when $G_{rz} = G_{\varphi z} = G$ the following bound can be obtained from formulae (24) and (26).

$$2.22222 \leq \frac{S}{Ga^4} \leq 2.25185. \tag{28}$$

4 NUMERICAL EXAMPLE

The following data are used in the numerical example

$$a = 0.035 \text{ m},$$
$$G_{rz} = 1 \times 10^{10} \text{ Pa}, \; G_{\varphi z} = 1.6 \times 10^{10} \text{Pa}.$$

Figures 2 and 3 show the graphs of the shearing stress $\tau_{\varphi z}(r,\varphi)$ as a function of r on the axis x and on the axis y.

$$\tilde{\tau}_{rz}(r,\varphi) = \frac{40\vartheta}{ka^2}(-2r^3\sin\varphi\cos\varphi + 4r^3\cos^3\varphi\sin\varphi) \tag{20}$$

$$\tilde{\tau}_{\varphi z}(r,\varphi) = -\frac{40\vartheta}{ka^2}(-2a^2r + 4r^3\cos^2\varphi - 4r^3\cos^4\varphi) \tag{21}$$

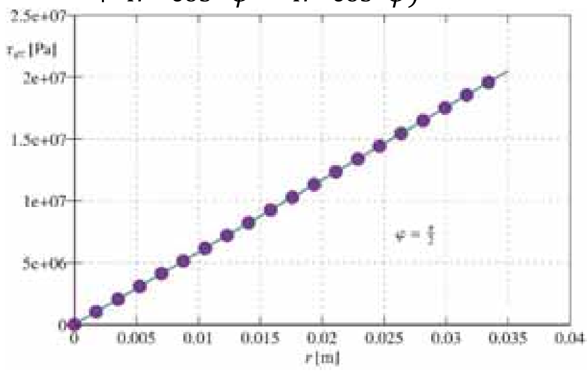

Figure 2 The plot of shearing stress $\tau_{\varphi z}$ as a function of r for $\varphi = 0$.

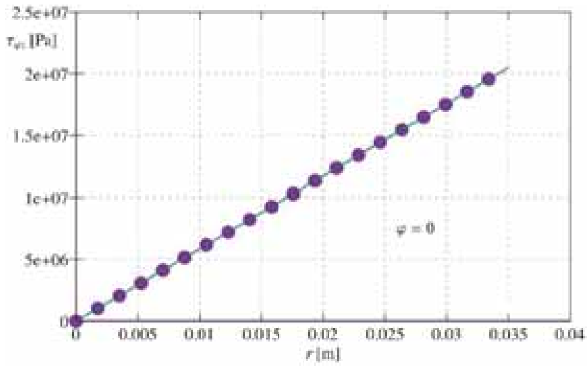

Figure 3 The plot of shearing stress$\tau_{\varphi z}$ as a function of r for $\varphi = \frac{\pi}{2}$.

In Figure 4 the plot of von Mises stress is presented for $x = a, -a \leq y \leq a$.

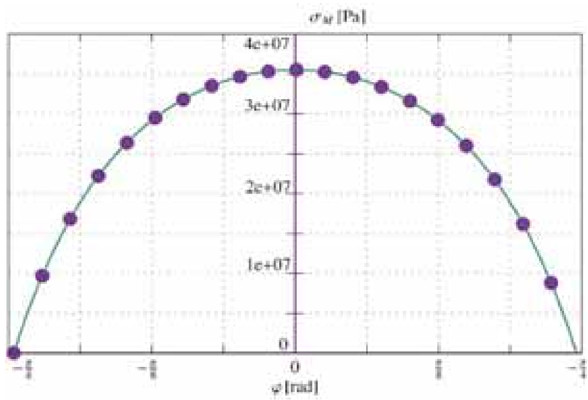

Figure 4 The plot of the graph of von Mises stress for $x = a$ and $|y| \leq a$.

The application of the bounding formula of torsional rigidity gives

$$S_L = 52072.45 \text{ Nm}^2 < S < S_U = 52608.05 \text{ Nm}^2 \tag{27}$$

Let $\overline{S}$ be defined as

$$\overline{S} = \frac{1}{2}(S_L + S_U). \quad (28)$$

The approximate value of relative error ε is as follows

$$\varepsilon = \frac{S_U - S_L}{\overline{S}} = 0.0102397 \quad (29)$$

that is 1.02397 %.

5 CONCLUSIONS

The Saint-Venant torsion of homogeneous, linearly elastic, cylindrical orthotropic solid bar with square cross section is analyzed. Approximate expressions are given for the shearing stresses. Two-side bound are formulated for the torsional rigidity. The application of derived formulae is illustrated by a numerical example. The obtained formulae can be used to check the validity of numerical results which are derived by other numerical methods such as FEM, finite differences etc.

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