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# DIRECT INTERPRETATION OF SELF-POTENTIAL ANOMALIES DUE TO SPHERICAL STRUCTURES — A HILBERT TRANSFORM TECHNIQUE

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A direct interpretation is developed by means of horizontal and vertical derivatives of self-potential anomalies due to point poles and spheres. The vertical derivative is obtained via the Hilbert transform. The depth to the centre of the sphere, the angle of polarization and the multiplicative factor comprising the resistivity of the surrounding medium and current density are evaluated directly by simple mathematical expressions based on the abscissae of the points of intersection of these derivatives. The procedure is illustrated with a theoretical example in each case. The effect of random noise on the interpretation is studied by adding Gaussian noise to the anomaly whereupon it was found that noise has little influence on the process of interpretation. Analysis of the field data pertaining to the 'Weiss' anomaly of eastern Turkey substantiates the validity of the method. This interpretive procedure can easily be automated.

**Keywords: self-potential, anomaly, convolution, Hilbert transform, spherical structure**

## 1. Introduction

Of all the electric methods, the use of the self-potential method enjoys wide application including engineering, ground water, subsurface temperature distribution as well as mineral investigations. Fast and refined techniques for interpreting self-potential anomalies are not in vogue: this is because self-potential data are complicated by a considerable amount of noise, and may be constant or varying. High noise levels pose serious interpretational hazards in developed areas. However, scrupulously performed field procedures combined

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with noise source mapping will certainly give reproducible data and prevent noise from being mistaken for signals.

In general the quantitative interpretation of SP anomalies is accomplished by approximating the source to simple regular geometrical shapes such as cylinders, spheres, sheets, etc. The available techniques are similar to those developed for gravity and magnetic interpretation. Despite there being quite a few methods for interpreting SP anomalies, they are subject to many constraints.

Estimates of anomaly source configuration, depth and other parameters for simple models can be made using analytical formulae. Some of these methods make use either of certain characteristic points on the anomaly or of nomograms [YÜNGÜL 1950]. The use of nomograms based on the method of BHATTACHARYA, ROY [1981], is somewhat crude and inadequate [RAJAN et al. 1986]. Curve matching techniques [MEISER 1962] proved to be cumbersome, especially when there are too many parameters to be determined. The least squares method involves a series of trials in minimizing the difference between observed and calculated values. All these methods have their own interpretational drawbacks.

AGARWAL [1985] made use of the amplitude of the analytic signal for interpreting self-potential anomalies caused by spherical structures. Despite the fact that this approach is essentially based on the use of the Hilbert transform, the method remains an empirical one wherein the parameters of the causative body are somewhat related to the shape and size of the amplitude curve [NABIGHIAN 1972].

We present herein a simple and refined mathematical procedure using the Hilbert transform for a straightforward evaluation of the parameters of the body. This method is based on the use of horizontal and vertical derivatives of the anomalous field. The vertical derivative is obtained via the Hilbert transform. The method is bound to yield more accurate results than the methods listed above since the present method is based on the real roots of the derivatives of the SP anomalies [SUNDARARAJAN et al. 1990, SUNDARARAJAN et al, 1994]. Similar methods are made use of in the gravity and magnetic interpretation and found to be much simpler as well as being elegant and accurate [MOHAN et al. 1982, SUNDARARAJAN 1982, SUNDARARAJAN et al. 1983].

If $HD(x)$ and $VD(x)$ are the horizontal and vertical derivatives respectively of any order of the self-potential anomaly then, according to SUNDARARAJAN [1982], they form a Hilbert transform, which implies that the vertical derivative of the field can be computed from the horizontal derivative or vice versa. This is expressed as:

$$HD(x) \longleftarrow\text{----------}HT\text{---------}\longrightarrow VD(x)$$

where $HT$ is the Hilbert transform operator.

This can mathematically be stated as:

$$VD(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{HD(y)}{(x-y)} \, dy \tag{1}$$

The divergence of $x=y$ is allowed for by taking Cauchy's principal value ($P$) of the integral [BRACEWELL 1986]. The function $VD(x)$ is a linear function of $HD(x)$ which in fact is obtainable from $HD(x)$ by convolution with $(1/\pi x)$. This relationship is stated as:

$$VD(x) = (1/\pi x) * HD(x) \tag{2}$$

where $*$ denotes the convolution.

The discrete form of the above relation is expressed as [TANER et al. 1979]:

$$VD(x) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} HD(x - n\Delta x) \frac{\sin^2(n\pi/2)}{n} \qquad n \neq 0 \tag{3}$$

where $\Delta x$ is the station interval and $n$ is the total number of stations on a profile.

Alternatively, the discrete Hilbert transform (DHT) can also be computed via the discrete Fourier transform (DFT) very efficiently using the fast Fourier transform (FFT) algorithm. The DHT as a function of DFT can be defined as [MOHAN et al. 1982]:

$$VD(n\Delta x) = \sum_{m=0}^{N-1} \text{Im } HD(mw_0) \cos(mw_0 n\Delta x) - \text{Re } HD(mw_0) \sin(mw_0 n\Delta x) \tag{4}$$

where Re $HD(mw_0)$ and Im $HD(mw_0)$ are real and imaginary components of the DFT of the horizontal derivative, $w_0$ is the fundamental frequency expressed in radian/unit length and is given as $w_0 = 2\pi/N\Delta x$, and $N$ is the total number of samples on a profile.

Further, the complex analytical signal $A(x)$ can be defined as a complex function whose real and imaginary parts are the horizontal and vertical derivatives of the potential function:

$$A(x) = HD(x) + iVD(x) \tag{5}$$

The amplitude of the analytical signal helps to locate the origin of the causative body. The amplitude is given as:

$$AA(x) = [HD(x)^2 + VD(x)^2]^{1/2} \tag{6}$$

The function $AA(x)$ attains its maximum over the origin. It is true for all two- and three-dimensional structures. In addition, the amplitude is also useful for interpreting structures of arbitrary shape. The location of origin based on amplitude is the unique feature of this method. The method remains the same for all potential field anomalies of 2-D and 3-D structures irrespective of their geometrical configuration.

## 2. SP field due to a point pole

*Figure 1* represents the geometry of the pole. Let $SP1(x)$ be the potential at a point $P(x, y = 0, 0)$ due to a point pole of strength $E$ placed at a point $Q(x', z)$. The potential due to such a pole is given as [AGARWAL 1985]:

$$SP1(x) = \frac{E}{[(x - x')^2 + z^2]^{1/2}} \tag{7}$$

where $z$ is the depth to the pole.

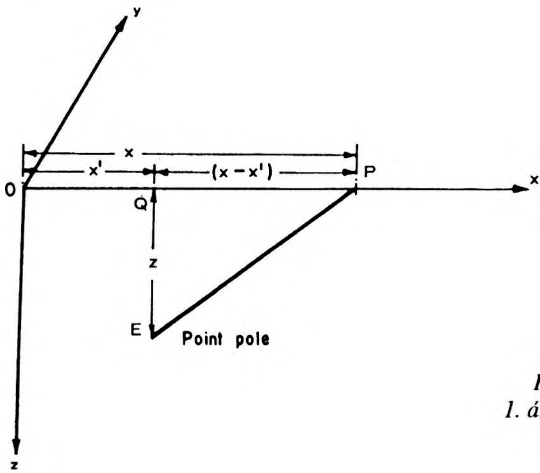


*Fig. 1.* Geometry of point pole
*1. ábra.* Pontszerű pólus geometriája

Performing partial differentiation of Eq. (7) with respect to $x$ and $z$, the horizontal and vertical derivatives of $SP1(x)$ are obtained as:

$$HD1(x) = \frac{-E\,(x - x')}{[(x - x')^2 + z^2]^{3/2}} \tag{8}$$

$$VD1(x) = \frac{-E\,z}{[(x - x')^2 + z^2]^{3/2}} \tag{9}$$

Since Eqs. (8) and (9) are of first degree in $x$, we have:

$$HD1(x) = VD1(x) \qquad \text{at} \qquad x = x_1$$

which implies that,

$$z = (x - x') = x_1 \tag{10}$$

i.e. the depth to the point pole is equal to the abscissa of the point of intersection of the horizontal and vertical derivatives.

Since the potential or the derivatives are known at every $x$, the pole strength $E$ can be calculated from Eqs. (7), (8) or (9).

## 3. SP field due to a sphere

With reference to *Fig. 2*, the geometry of the obliquely polarized sphere with radius $a$ is considered. In the cartesian coordinate system, $O$ is the origin: on the surface at a point vertically above the centre of the sphere. The axis of the sphere is parallel to the $y$-axis. $AA'$ is the axis of polarization, $\Theta$ is included between the polarization- and $x$-axis. $P$ is the point of observation at a distance $x$ from the origin, $\alpha$ is the angle between the axis of polarization and the line passing through the centre of the sphere and $P$. $Q$ is the point where the potential is zero. Therefore, the potential at a point $P$ on the surface is given as [AGARWAL 1985]:

$$SP2(x) = C\frac{(z\cos\Theta + x\sin\Theta)}{(x^2+z^2)^{3/2}} \tag{11}$$

where $z$ is the depth to the centre of the sphere, $\Theta$ is the angle of polarization, and $C$ is a constant comprising the current density ($I$) and the resistivity ($\rho$) of the surrounding medium as:

$$C = I\rho/2\pi .$$

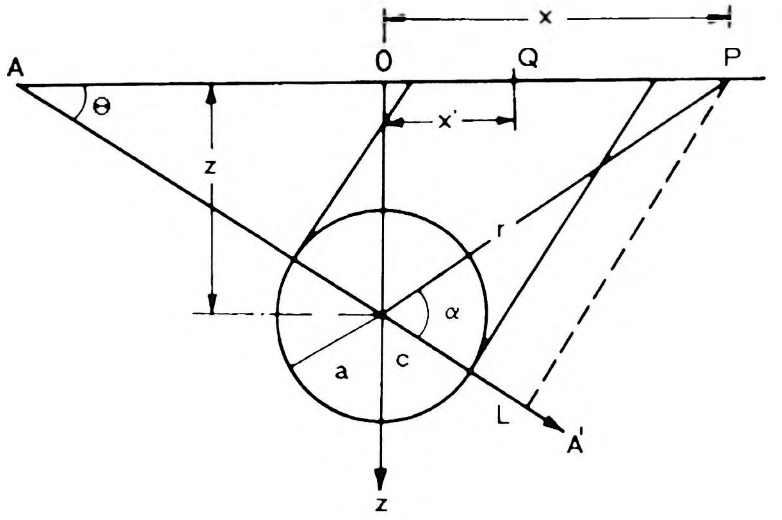


*Fig. 2.* Geometry of sphere
*2. ábra.* Gömb alakú pólus geometriája

Partial differentiation of Eq. (11) with respect to $x$ and $z$ yields the horizontal and vertical derivatives of $SP2(x)$:

$$HD2(x) = C\,\frac{(z^2 - 2x^2)\sin\Theta - 3xz\cos\Theta}{(x^2 + z^2)^{5/2}} \tag{12}$$

and

$$VD2(x) = C\,\frac{(x^2 - 2z^2)\cos\Theta - 3xz\sin\Theta}{(x^2 + z^2)^{5/2}}. \tag{13}$$

Analyzing the results we obtain that at $x$=0, Eqs. (12) and (13) reduce to:

$$HD2(0) = C\sin\Theta/z^3 \tag{14}$$

$$VD2(0) = -2C\cos\Theta/z^3 \tag{15}$$

The angle of polarization $\Theta$ can be evaluated from Eqs. (14) and (15) as:

$$\Theta = \tan^{-1}[-2HD2(0)/VD2(0)] \tag{16}$$

Since Eqs. (12) and (13) are of second degree in $x$, we have:

$$HD2(x) = VD2(x) \qquad \text{at} \qquad x = x_1, x_2 \tag{17}$$

Further simplication leads to the solution of $z$ as:

$$z = (x_1 + x_2)\,\frac{(\cos\Theta + 2\sin\Theta)}{3(\sin\Theta - \cos\Theta)} \tag{18}$$

It would be worth mentioning here that $z$ tends to infinity when $\Theta$=45°; this is purely a hypothetical case and it can be attributed to the fact that $(x_1+x_2)=0$ which introduces catastrophe in the mathematical procedure. That is, the magnitude of the real roots of the derivatives are equal with opposite sign. In this case, which is seldom encountered in practice, the depth is simplified as:

$$z = x_1 = -x_2 \tag{19}$$

Also, from Eqs. (12) and (13), the constant term $C$ is obtained as:

$$C = \frac{2z^3[HD2(0)^2 + VD2(0)^2]^{1/2}}{(1 + 3\cos^2\Theta)} \tag{20}$$

Thus, equation (20) yields either the current density ($I$) or the resistivity ($\rho$) of the surrounding medium provided the other parameters are known.

## 4. Theoretical example

The interpretive process detailed above is illustrated with a theoretical example in each case. The self-potential anomalies (*Figs. 3* and *4*) pertaining to point pole and sphere are computed using Eqs. (7) and (11) for a set of model parameters (*Table I*). *Figures 5* and *6* correspond to the first horizontal

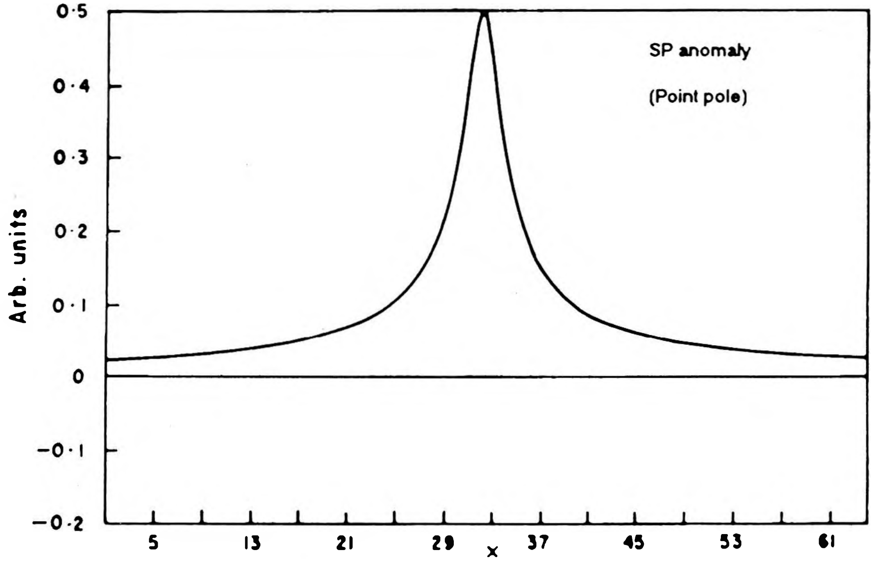


*Fig. 3.* SP anomaly due to a point pole
*3. ábra.* Pontszerű pólus által keltett SP anomália

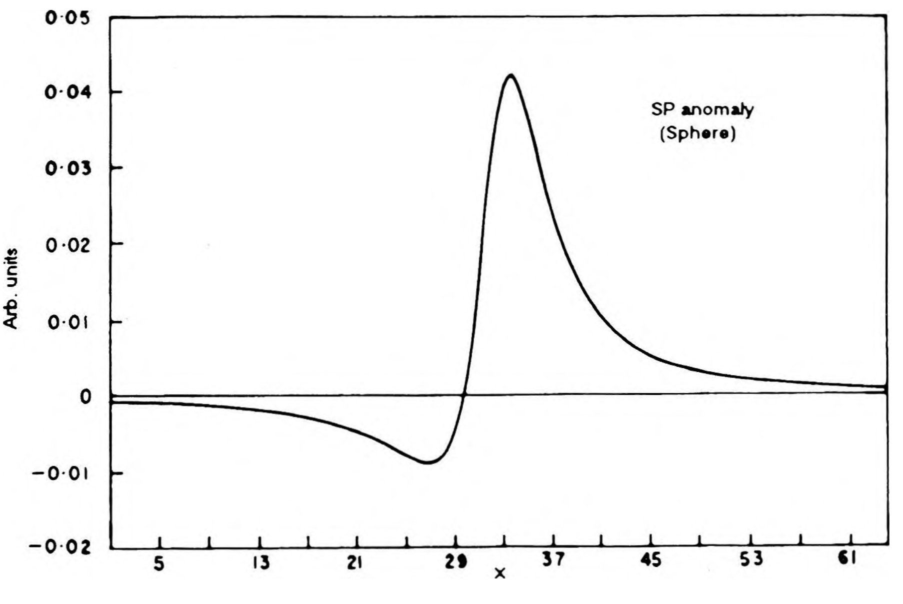


*Fig. 4.* SP anomaly due to a sphere
*4. ábra.* Gömb alakú pólus által keltett SP anomália

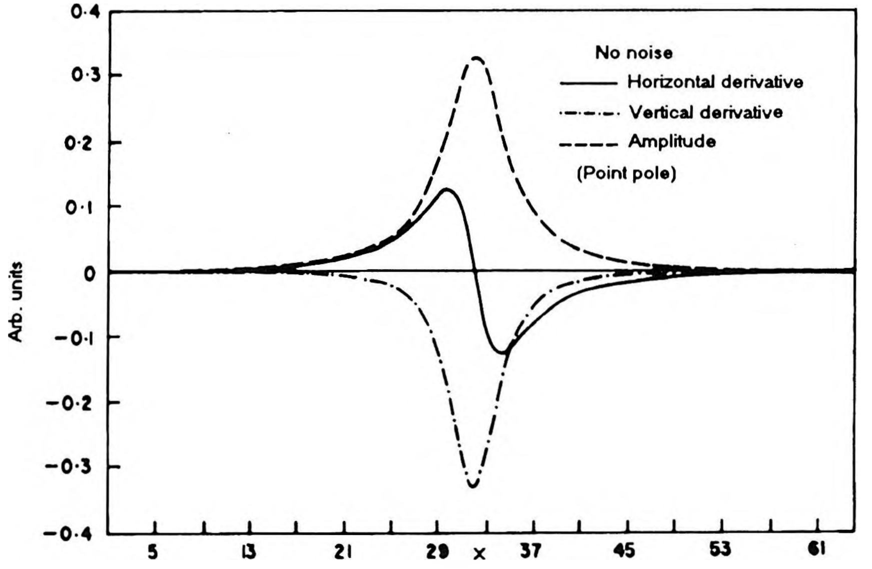


*Fig.* 5. First horizontal derivative, vertical derivative and amplitude of point pole
5. *ábra.* Pontszerű pólus első horizontális deriváltja, vertikális deriváltja és amplitúdója

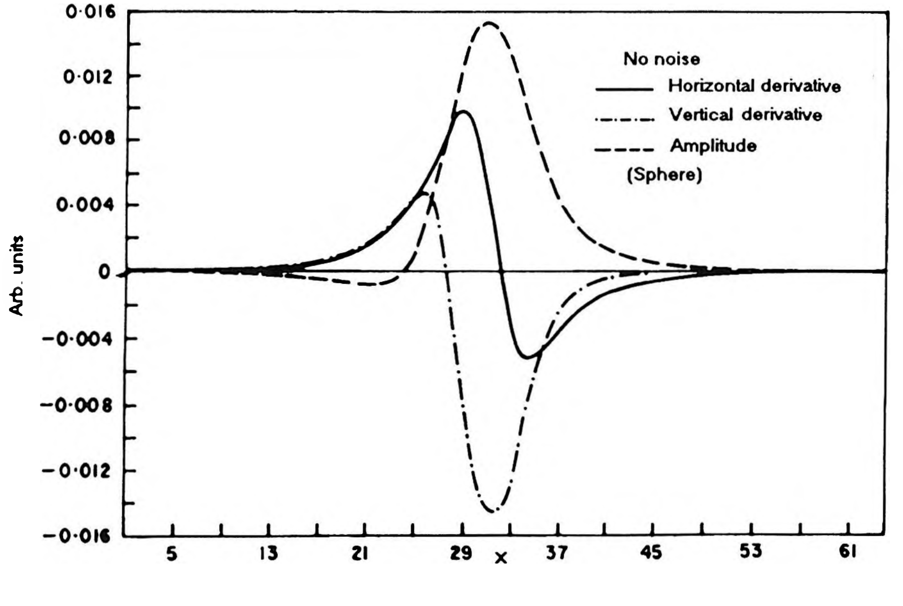


*Fig.* 6. First horizontal derivative, vertical derivative and amplitude of sphere
6. *ábra.* Gömb alakú pólus első horizontlis deriváltja, vertikális deriváltja és amplitúdója

derivative, the Hilbert transform and the amplitude in the case of point pole and sphere respectively. After precise location of origin with a knowledge of amplitude, the parameters of the causative bodies are evaluated based on the procedure detailed in the text. The assumed and interpreted values of the parameters are given in Table I and are in very good agreement.

| Models | Parameters | Θ [degree] | $h^*$ | $C^*$ |
|---|---|---|---|---|
| Point pole | Assumed | - | 1.50 | 0.75 |
| | Interpreted (noise free) | - | 1.55 | 0.79 |
| | Interpreted (with noise) | - | 1.41 | 0.86 |
| | Error | - | 6 % | 14.66 % |
| Sphere | Assumed | 60.00 | 4.00 | 1.00 |
| | Interpreted (noise free) | 60.10 | 3.96 | 1.15 |
| | Interpreted (with noise) | 62.15 | 3.60 | 1.14 |
| | Error | 3.5 % | 10 % | 14 % |

*Table I.* Theoretical examples (* in arbitrary units)
*I. táblázat.* Elméleti példák (* tetszőleges egységekben)

## 5. Noise analysis

The effect of random noise on the interpretation is studied by incorporating Gaussian noise (*Fig. 7*) with SP anomalies. A part of this noise is added separately to the SP anomaly in both cases depending upon the magnitude of the SP field. The noisy SP anomalies due to these structures are shown in *Figs. 8* and *9* along with the noise free anomalies. The first horizontal derivative of these anomalies is computed by means of numerical differentiation, then their Hilbert transforms are obtained by a discrete convolution process and thereby the amplitudes are computed. *Figures 10* and *11* show the horizontal derivative, the Hilbert transform and the amplitude in the case of point pole and sphere respectively. As discussed earlier, the origin is located based on the amplitude information and the interpretation is carried out. The results are presented in Table I and it is observed that they showed no appreciable change due to the presence of noise in the SP anomalies.

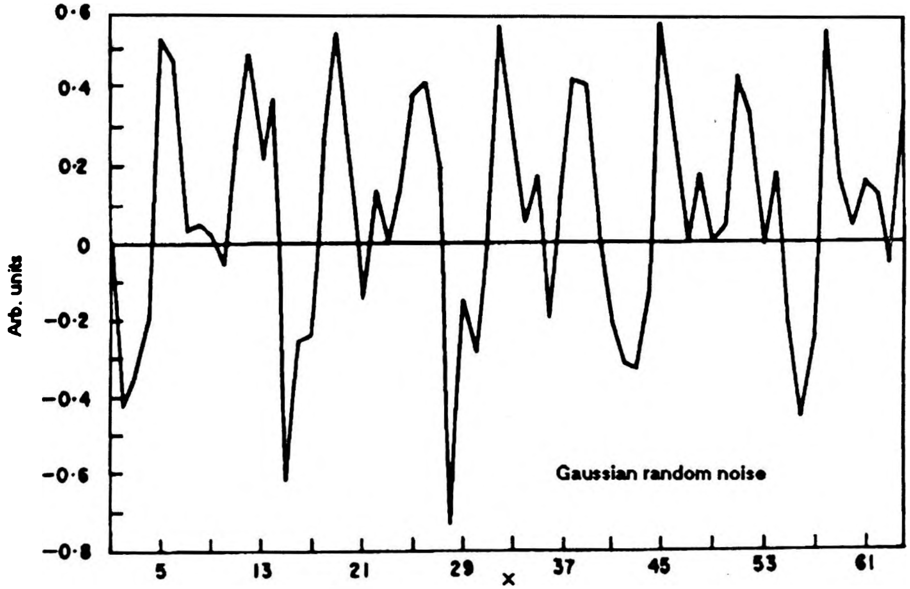


*Fig. 7.* Gaussian random noise
*7. ábra.* Gauss eloszlású véletlen zaj

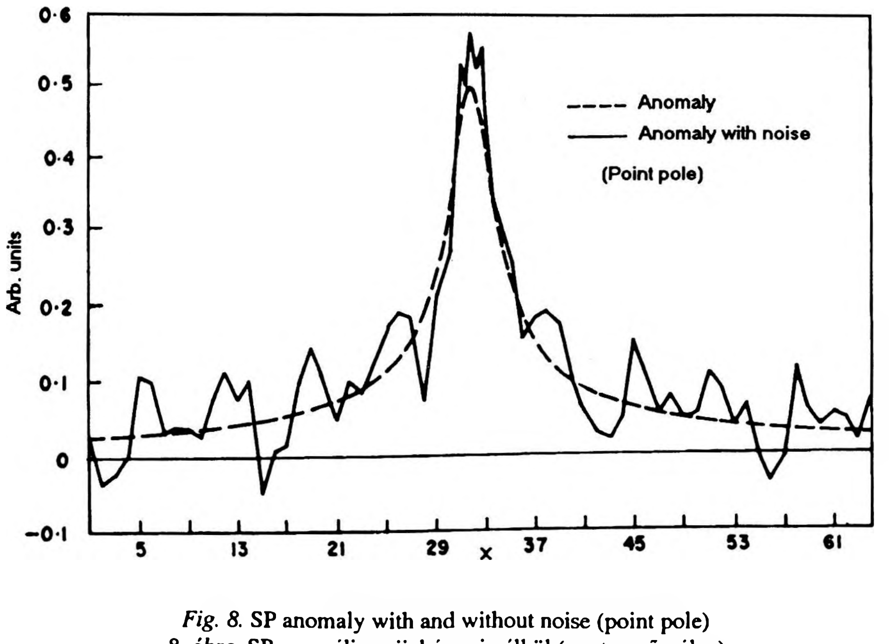


*Fig. 8.* SP anomaly with and without noise (point pole)
*8. ábra.* SP anomália zajjal és zaj nélkül (pontszerű pólus)

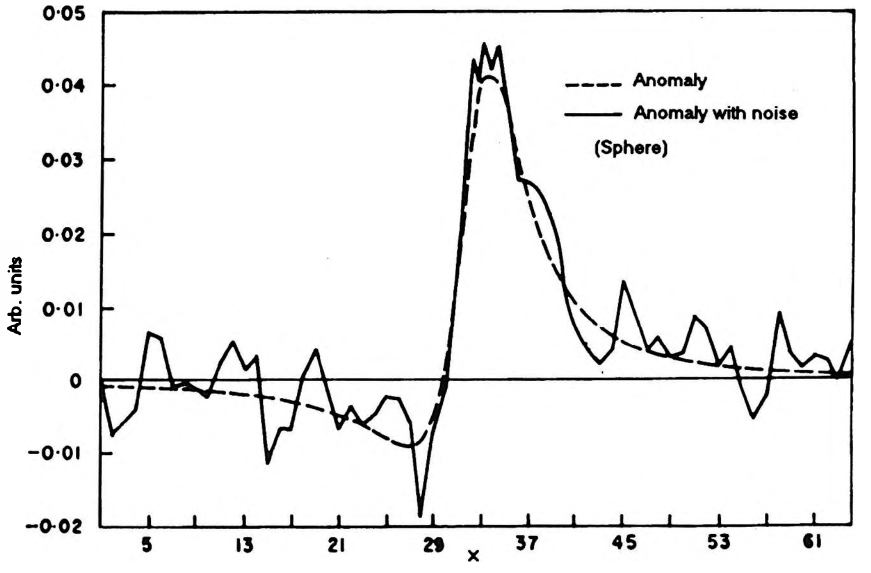


*Fig. 9.* SP anomaly with and without noise (sphere)
*9. ábra.* SP anomália zajjal és zaj nélkül (gömb alakú pólus)

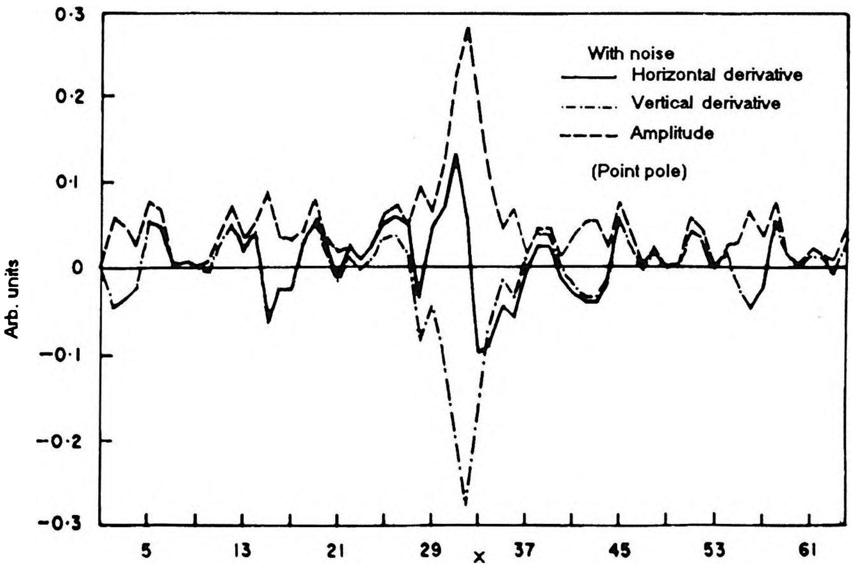


*Fig. 10.* First horizontal derivative of the noisy SP anomaly, the vertical derivative and the amplitude due to a point pole
*10. ábra.* Zajjal terhelt SP anomália első horizontális deriváltja, vertikális deriváltja és az amplitúdó, pontszerű pólus esetében

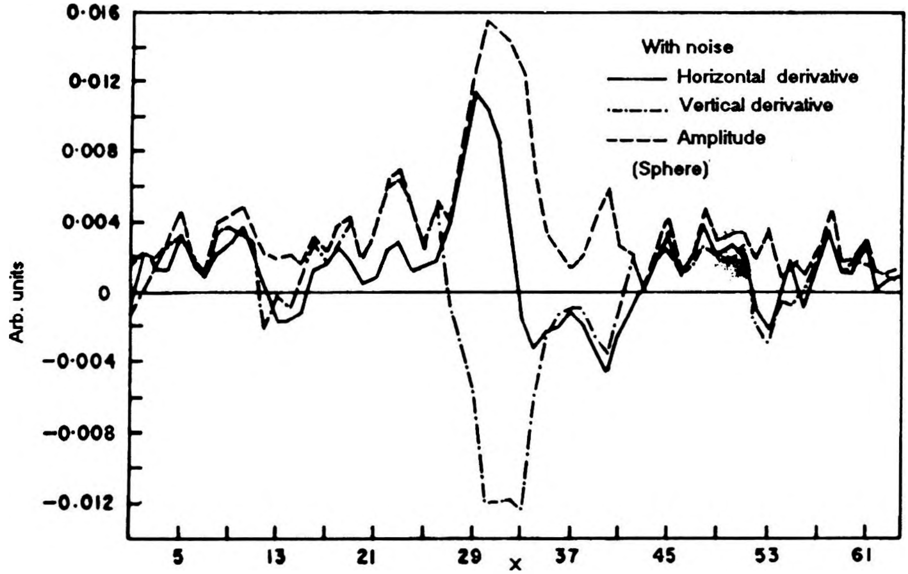


*Fig. 11.* First horizontal derivative of the noisy SP anomaly, the vertical derivative and the amplitude due to a sphere

*11. ábra.* Zajjal terheltt SP anomália első horizontális deriváltja, vertikális deriváltja és az amplitudó, gömb alakú pólus esetében

## 6. Field example

The procedure just described is exemplified by the well known 'Weiss anomaly' (*Fig. 12*) of the copper deposit in eastern Turkey [BHATTACHARYA, ROY 1981]. This anomaly is one kilometer northwest of the Madam copper mine and is assumed to be due to spherical structure. At an appropriate scale the entire anomaly is digitized and then the first horizontal derivative is computed by means of numerical differentiation. Then the vertical derivative is obtained using the Hilbert transform. The horizontal derivative, the vertical derivative and the amplitude are shown in *Fig. 13*. The parameters are evaluated based on the procedure discussed above and the results are compared with those of YÜNGÜL [1950] and BHATTACHARYA, ROY [1981] and are presented in *Table II.*

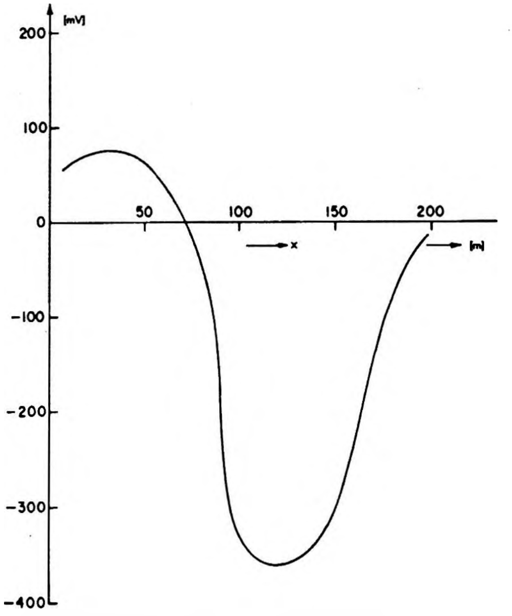


*Fig. 12.* SP anomaly (Weiss) of the Ergani copper deposit in eastern Turkey
*12. ábra.* Az Ergani rézelőfordulás (Weiss) SP-anomáliája, Kelet-Törökország

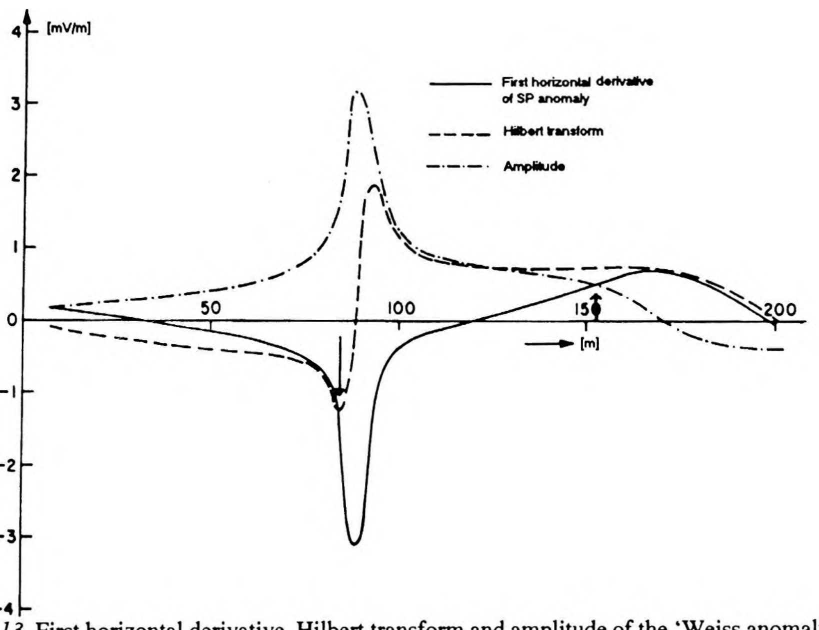


*Fig. 13.* First horizontal derivative, Hilbert transform and amplitude of the 'Weiss anomaly'
*13. ábra.* A „Weiss" anomália első horizontális deriváltja, Hilbert transzformáltja és amplitudója

| Parameters | Θ [degree] | z [meter] |
|---|---|---|
| Present method | 79.00 | 52.30 |
| YÜNGÜL [1950] | 64.00 | 53.80 |
| BHATTACHARYA, ROY [1981] | 54.00 | 30.00 |

*Table II.* Field example (Weiss anomaly)
*II. táblázat.* Terepi példa (Weiss anomália)

## 7. Discussion

It is observed from the results (Table I) of noisy and noise free anomalies that there is no drastic change due to the presence of low level noise. However, there is a difference of around 5% to 15% between the assumed and interpreted (with noise) values. The maximum difference is seen only in the constant term comprising the resistivity of the surrounding medium and the current density.

Thus, the results testify that the method has no appreciable effect on the presence of random noise in the SP anomalies. A similar trend is observed even when the noise level is increased. However it is to be noted that for a very large amount of noise, the origin is slightly shifted either to the left or right of the actual origin without altering the values of the abscissae of the points of intersection of the derivatives.

A degree of error is inevitable in any method that makes use of discrete analysis and perhaps the error can be minimized by choosing an optimum sampling interval for processing. Further, the precise location of origin together with the exact values of the abscissae of the point of intersection of the derivatives ensure better accuracy of the interpreted results. The amplitude is not only useful in locating the origin but can also be made use of in estimating the parameters particularly when the causative bodies are of arbitrary structure [NABIGHIAN 1972].

Thus, the various parameters of the causative body are obtained by simple mathematical expressions as functions of real roots of the derivatives of SP anomalies and hence the method can easily be automated. Therefore the method is practicable and can be recommended to practising geophysicists.

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# GÖMB ALAKÚ SZERKEZETEK ÁLTAL KELTETT SAJÁTPOTENCIÁL ANOMÁLIÁK KÖZVETLEN ÉRTELMEZÉSE — EGY HILBERT TRANSZFORMÁCIÓS ELJÁRÁS

N. SUNDARARAJAN és M. NARASIMHA CHARY

Közvetlen értelmezési módszert dolgoztak ki pontszerű pólusok és gömbök okozta sajátpotenciál anomáliák horizontális és vertikális deriváltjainak alkalmazásával. A vertikális derivált Hilbert transzformációval nyerhető. Ezen deriváltak metszéspontjainak abszcisszáin alapuló, egyszerű matematikai kifejezésekkel becsülik a gömb középpontjának felszíntől mért távolságát, a polarizációs szöget, és azt a szorzótényezőt, mely magába foglalja a környező közeg fajlagos ellenállását és az áram sűrűséget. Az eljárást minden esetben elméleti példával illusztrálják. A véletleneloszlású zaj értelmezésre gyakorolt hatását az anomáliához Gauss-eloszlású zajt adva tanulmányozzák és megállapítják, hogy a zajnak csekély hatása van az értelmezésre. A kelet-törökországi „Weiss" anomália terepi adatainak elemzése igazolja a módszer érévnyességét. Ez az értelmezési eljárás könnyen automatizálható.