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FAST COMPUTING OF TRANSIENT ELECTROMAGNETIC FIELD ON THE SURFACE OF A LAYERED HALF-SPACE

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Time domain electromagnetic fields can be computed by the spectral technique, viz. using the inverse Fourier-transform applied to frequency domain, or by solving Maxwell's equations in the time domain. For a layered half-space and for the total time domain, accurate computations for the latter method are very time consuming because of the presence of a partial differential equation in the derivation of the formulae determining the transient field that cannot be solved by analytical functions. The numerical solution of partial differential equations is very time consuming. On the other hand, the differential equation that arises in the frequency domain can be expressed by analytical functions. If we do not require an accurate solution to the total time domain, then the solution of the partial differential equation occurring in the time domain can be computed by analytical functions, too. The paper discusses a case which is valid for a non-conducting basement and is based on an asymptotic solution that is valid at late times.

Keywords: transient methods, electomagnetic field, half-space, dipole, computer programs

1. Introduction

In most cases the interpretation of the transient and other electromagnetic measurements is based on assuming a layered half-space at the site of the measurement and we try to determine the parameters of the half-space, from which a conclusion can be drawn on the geoelectric structure. In order to determine the layer parameters that belong to the measured curves a direct problem solving program is needed for computing the theoretical curves from optional layer parameters. Though in the case of the transient method there are many computational methods for solving the problem,

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none of them is so fast that it would not be worth the effort of increasing the speed of computation. The speed is especially important when applying a curve computating program for curve fitting. On a 12 MHz IBM PC/AT equipped with a coprocessor the computation of the direct problem by the spectral method takes about 10 s/layer and it gives accurate results at early and at late times [PRÁCSER 1986]. The transient electromagnetic field takes the form of a Hankel transform of a kernel function as a result of the solution of the Maxwell's equations in the time domain. This kernel function is the solution of a partial differential equation that can be solved by the finite difference method [GOLDMAN 1983]. Computation based on this theory is much slower than the spectral technique. If we accept that for the early times we do not get accurate transient field values then the kernel function in the time domain can be generated in the form of a series expansion. In that case the computation requires much less time than by the finite difference method. In the Soviet literature we find that this method used to be applied to two- and three-layered halfspaces [TIKHONOV, SKUGAREVSKAYA 1950]. At late times even the first term of the series gives an accurate result; at early times we have to take more terms into consideration. Even though this method was known as far back as 1950, at that stage of computational techniques it probably could not be applied in practice and once computers had come into general use in geophysics the spectral method was preferred in transient calculus. In the following we show the computation based on the series expansion of the kernel function for an n-layered model.

2. Transient field of an electric dipole at the surface of a layered half-space

Let us examine the electric dipole on the surface of a layered half-space when the current flowing in a conductor of elementary length is turned off at time t=0 and the effect of the displacement current is negligible. The induced electromagnetic field is defined by Maxwell's equations:

$$\text{rot}\,\vec{H} = \sigma\,\vec{E} + \vec{J} \qquad \text{div}\,\vec{H} = 0 \tag{1}$$

$$\text{rot}\,\vec{E} = -\mu\,\frac{\partial}{\partial t}\,\vec{H} \qquad \text{div}\,\vec{E} = 0$$

where:

$\vec{E}$ — electric field vector

$\vec{H}$ — magnetic field vector

μ — magnetic susceptibility
t — time
$\vec{J} = (J_x, 0, 0), \quad J_x = \dfrac{I(t)\, dl\, \delta(z)\, \delta(r)}{2\pi r}$ — exciting current
r — length between the dipole and the measuring point

$$I(t) = \begin{cases} I_0 & t \le 0 \\ 0 & t > 0 \end{cases}$$ — current flowing in the dipole

dl — length of the electromagnetic dipole
$\delta(z)$, $\delta(r)$ — Dirac-delta functions
σ — conductivity function depending only on the z coordinate that is defined as follows:

$$\sigma(z) = \begin{cases} 0 & z \ge z_0 = 0 \\ \sigma_j & z_{j-1} > z > z_j \qquad j = 1, \ldots, n-1 \\ \sigma_n = 0 & z < z_{n-1} \end{cases}$$

z_j — coordinates of the layer boundaries
n — number of layers

We shall briefly discuss the most important steps of the derivation of Eq. (1) that defines the transient field. As can be seen from the definition of σ we deal only with that case when the conductivity of the n-th layer is zero. One of the disadvantages of the computation based on the theory in question is that the n-th layer has to be a non-conductor or an ideal conductor. Here we discuss only the case of $\sigma_n = 0$ because it has much greater importance in practice than the case of $\sigma_n = \infty$. With the spectral technique there is no need for such a restriction. We start to solve the system of equations (1) in the usual way by introducing the $\vec{A}$ vector potential.

$$\vec{H} = \operatorname{rot} \vec{A}, \qquad \vec{A} = (A_x, 0, A_z) \tag{2}$$

The $\vec{A}$ vector potential, as can be derived from Maxwell's equations, satisfies the following partial differential equation:

$$\vec{A} - \mu\sigma \frac{\partial}{\partial t} \vec{A} = -\vec{J} \tag{3}$$

Bearing in mind that in most cases in transient measurement only the vertical component of the time derivative of the magnetic field vector is measured it is enough to determine the horizontal component of the vector potential because H_z does not depend on A_z (Eq. 2). From Eq. (3), written in cylindrical coordinates, the integral representation of A_x can be derived by the separation of variables:

$$A_x = c \int_0^\infty J_0(\lambda r)\, X(\lambda,z,t)\, d\lambda$$

where:

J_0 — Bessel function of the first kind, zero order

$X(\lambda,z,t)$ — kernel function in time domain

Constant c can be obtained if we compare the magnetic field of an electric dipole in vacuum, expressed by vector potential, with the law of Biot-Savart, by which we get:

$$c = \frac{I_0\, dl}{4\pi}$$

So the formula defining the x component of the vector potential is:

$$A_x = \frac{I_0\, dl}{4\pi} \int_0^\infty J_0(\lambda r)\, X(\lambda,z,t)\, d\lambda \tag{4}$$

3. Determination of the kernel function $X(\lambda,z,t)$

From Eq. (3) it follows that the $X(\lambda,z,t,)$ kernel function satisfies the following partial differential equation:

$$\frac{\partial^2 X(\lambda,z,t)}{\partial z^2} - \lambda^2 X(\lambda,z,t) = \mu\sigma \frac{\partial}{\partial t} X(\lambda,z,t) \tag{5}$$

For the uniqueness of Eq. (5) it is necessary to satisfy certain boundary conditions. The electromagnetic components and their time derivatives have a continuous transition through layer boundaries. For $t > 0$ it is valid for the kernel function $X(\lambda,z,t)$ and for its time derivatives too. If we take it into consideration that for $z > z_0$ and $z < z_{n-1}$ the conductivity of the medium is equal to zero then from Eq. (5) it follows that the kernel function $X(\lambda,z,t)$ has the form of $c_0\, e^{-\lambda z}$ and $c_n\, e^{\lambda z}$ where c_0 and c_n are constants. So at the uppermost and at the lowest layer boundaries Eq. (5) has to be complemented with the following boundary conditions:

$$\frac{\partial X(\lambda,z,t)}{\partial z} + \lambda X(\lambda,z,t) = 0 \qquad z = z_0 \tag{6a}$$

$$\frac{\partial X(\lambda,z,t)}{\partial z} - \lambda X(\lambda,z,t) = 0 \qquad z = z_{n-1} \tag{6b}$$

For $t \le 0$ current flows at a constant I_0 current intensity, viz. the time derivative is equal to zero in Eq. (5). Because of this the A_x component of the vector potential is the same as it should be in a vacuum from which we get the boundary condition at t=0:

$$X(\lambda,z,t) = e^{\lambda z} \qquad t = 0, \quad z < 0 \tag{7}$$

Partial differential equation (5) defining the kernel function $X(\lambda,z,t)$ can be solved by the finite difference method, too [GOLDMAN 1983] but this method is very time consuming. Here we would comment that the frequency domain form of Eq. (5) will be simpler because instead of time derivation there is conjunction $i\omega$, where $i = \sqrt{-1}$ is the imaginary unit and ω is the radian frequency. So the exact solution can be expressed by a linear combination of exponential functions.

In this paper the solution of Eq. (5) is sought by separating the variables in the form of an infinite series:

$$X(\lambda,z,t) = \sum_{k=1}^{\infty} \beta_{k,j}(\lambda,z)\, \alpha_{k,j}(t) \qquad j = 1,\ldots, n-1 \tag{8}$$

If the expansion in series (8) of kernel function $X(\lambda,z,t)$ is substituted into Eq. (5) we get the differential equation defining functions $\beta_{k,j}$ and $\alpha_{k,j}$ in the j-th layer:

$$\frac{\frac{\partial^2}{\partial z^2}\beta_{k,j}(\lambda,z)}{\beta_{k,j}(\lambda,z)} - \lambda^2 = -\nu_{k,j} \tag{9}$$

$$\mu\sigma_j \frac{\frac{\partial}{\partial t}\alpha_{k,j}(t)}{\alpha_{k,j}(t)} = -\nu_{k,j} \tag{10}$$

The solution of Eq. (10) is:

$$\alpha_{k,j}(t) = c_k\, e^{-\frac{\nu_{k,j}}{\mu\sigma_j} t} \tag{11}$$

From the continuity of functions $\alpha_{k,j}$ in direction z it follows that the exponents have to be independent of j. Because of this the $\nu_{k,j}$ separational constants can be determined as a function of conductivity:

$$\nu_{k,j} = w_j \chi_k$$

where $w_j = \frac{\sigma_j}{\sigma_1}$, and χ_k is a constant that will be determined later.

Knowing $\nu_{k,j}$ let us rearrange differential equation (9) determining $\beta_{k,j}$ in the j-th layer:

$$\frac{\partial^2}{\partial z^2} \beta_{k,j}(\lambda,z) = -\beta_{k,j}(\lambda,z)\,(w_j \chi_k - \lambda^2) \qquad z_{j-1} > z > z_j \tag{12}$$

Depending on the sign of the $\omega_j \chi_k - \lambda^2$ function, $\beta_{k,j}$ is a linear combination of trigonometric or hyperbolic functions:

$$\beta_{k,j}(\lambda,z) = a_{k,j} \cos P_{k,j}(z - z_{j-1}) + b_{k,j} \sin P_{k,j}(z - z_{j-1})$$

$$P_{k,j} = \sqrt{w_j \chi_k - \lambda^2} \qquad \text{if} \quad w_j \chi_k > \lambda^2 \tag{13a}$$

$$\beta_{k,j}(\lambda,z) = a_{k,j}\, \mathrm{ch}\, P_{k,j}(z - z_{j-1}) + b_{k,j}\, \mathrm{sh}\, P_{k,j}(z - z_{j-1})$$

$$P_{k,j} = \sqrt{\lambda^2 - w_j \chi_k} \qquad \text{if} \quad \lambda^2 > w_j \chi_k \tag{13b}$$

The values of χ_k have to be chosen in such a way that the kernel function $X(\lambda,z,t)$ and its derivative in direction z should be continuous through the layer boundaries and satisfy Eq. (6). According to the series expansion (8) functions $\beta_{k,j}$ have to satisfy the same conditions. In consequence of Eq. (6a) at the surface:

$$\lambda\, a_{k,1} + P_{k,1}\, b_{k,1} = 0 \tag{14a}$$

At the j-th layer boundary in consequence of the continuity of $\beta_{k,j}$:

$$a_{k,j} \cos \Phi_{k,j} + b_{k,j} \sin \Phi_{k,j} - a_{k,j+1} = 0 \tag{14b}$$

At the j-th layer boundary in consequence of the continuity of $\frac{\partial}{\partial z} \beta_{k,j}$:

$$-a_{k,j} P_{k,j} \sin \Phi_{k,j} + b_{k,j} P_{k,j} \cos \Phi_{k,j} - b_{k,j+1} P_{k,j+1} = 0 \tag{14c}$$

At the lowest layer boundary in consequence of Eq. (6b):

$$-a_{k,n-1} P_{k,n-1} \sin \Phi_{k,n-1} - a_{k,n-1}\lambda \cos \Phi_{k,n-1} +$$
$$+ b_{k,n-1} P_{k,n-1} \cos \Phi_{k,n-1} - b_{k,n-1} \lambda \sin \Phi_{k,n-1} = 0 \qquad (14d)$$

where:

$$\Phi_{k,j} = P_{k,j}(z_j - z_{j-1}) \qquad j = 1,\ldots,n-1$$

Equations (14) are valid in the case of $w_j \lambda_k > \lambda^2$ in every layer, viz. $\beta_{k,j}(\lambda,z)$ is generated by linear combination of sine and cosine functions (13a). Formulae similar to Eqs. (14) are valid even if function $\beta_{k,j}(\lambda,z)$ is a linear combination of functions that are hyperbolic in one layer or some layers only (13b). Thus coefficients $a_{k,j}$ and $b_{k,j}$ are determined by the homogeneous system of linear equations (14). In the case of n layers the number of equations and unknowns is $2(n$-1). It is necessary that the determinant of the system of equations be zero in order to have a solution of this system in addition to the solution that is identical with zero. This can be achieved by choosing χ_k properly. Bearing in mind that only the proximal elements of the main diagonal are not equal to zero, the value of the determinant can be computed by a relatively simple algorithm. Hereafter χ will be marked with index k occurring in the expansion of Eq. (8) when it de facto indicates a number for which the system of equations (14) can be solved. Let us show as an example the determinant of the system of equations (14) for a four-layer case for every χ where condition $\chi w_j - \lambda^2 > 0$ is satisfied in every layer:

$$D(\lambda,\chi) =$$

$$\begin{vmatrix} \lambda, & P_1, & 0, & 0, & 0, & 0, \\ \cos\Phi_1, & \sin\Phi_1, & -1, & 0, & 0, & 0, \\ -P_1\sin\Phi_1, & P_1\cos\Phi_1, & 0, & -P_2, & 0, & 0, \\ 0, & 0, & \cos\Phi_2, & \sin\Phi_2, & -1, & 0, \\ 0, & 0, & -P_2\sin\Phi_2, & P_2\cos\Phi_2, & 0, & -P_3 \\ 0, & 0, & 0, & 0, & -P_3\sin\Phi_3-\lambda\cos\Phi_3, & P_3\cos\Phi_3-\lambda\sin\Phi_3 \end{vmatrix}$$

where:

$$P_j = \sqrt{\chi w_j - \lambda^2}, \qquad \Phi_j = (z_j - z_{j-1})\, P_j$$

Now we show the determinant-computing algorithm taking into account the case of both Eq. (13a) and Eq. (13b). Let $D_{j,j}$ and $D_{j+1,j}$ denote minor determinants. $D_{j,j}$ consists of the first j lines and columns of $D(\lambda,\chi)$ but in the case of $D_{j+1,j}$ instead of j-th lines of $D(\lambda,\chi)$, $(j+1)$-th lines will occur. Let β_j denote the solution of the system of equations (12) in the j-th layer for arbitrary χ. The minor determinants that correspond to the first layer are:

$$D_{2,2}=\lambda \sin\Phi_1 - P_1\cos\Phi_1$$

$$D_{3,2} = \lambda P_1 \cos\Phi_1 + P_1^2 \sin\Phi_1 \tag{15a}$$

Or if $\beta_1(\lambda,z)$ is generated by hyperbolic functions:

$$D_{2,2}=\lambda \,\mathrm{sh}\,\Phi_1 - P_1\mathrm{ch}\,\Phi_1$$

$$D_{3,2}=\lambda P_1\mathrm{ch}\,\Phi_1 - P_1^2\mathrm{sh}\,\Phi_1 \tag{15b}$$

Computation of minor determinants belonging to the $(j+1)$-st layer on the basis of the minor determinants of the j-th layer is carried out in the following way:

$$D_{2(j+1),\,2(j+1)} = D_{2j,\,2j}\, P_{j+1} \cos\Phi_{j+1} + D_{2j+1,\,2j} \sin\Phi_{j+1}$$

$$D_{2(j+1)+1,\,2(j+1)} = -D_{2j,\,2j}\, P_{j+1}^2 \sin\Phi_{j+1} + D_{2j+1,\,2j} \cos\Phi_{j+1} \tag{16a}$$

The same expression when $\beta_{j+1}(\lambda,z)$ is a linear combination of hyperbolic functions follows:

$$D_{2(j+1),\,2(j+1)} = D_{2j,\,2j}\, P_{j+1}\, \mathrm{ch}\,\Phi_{j+1} + D_{2j+1,\,2j}\, \mathrm{sh}\,\Phi_{j+1}$$

$$D_{2(j+1)+1,\,2(j+1)} = D_{2j,\,2j}\, P_{j+1}^2\, \mathrm{sh}\,\Phi_{j+1} + D_{2j+1,\,2j}\, \mathrm{ch}\,\Phi_{j+1} \tag{16b}$$

Finally the total determinant based on the minor determinant corresponding to the $(n$-2)-nd layer is:

$$D(\lambda,\chi) = D_{2(n-2),\,2(n-2)}\,(-P_{n-1}^2 \sin\Phi_{n-1} - P_{n-1}\, \lambda \cos\Phi_{n-1}) +$$

$$+ D_{2(n-2)+1,\,2(n-2)}\,(P_{n-1} \cos\Phi_{n-1} - \lambda \sin\Phi_{n-1}) \tag{17a}$$

If $\beta_{n-1}(\lambda,z)$ is a linear combination of hyperbolic functions the same expression will be:

$$D(\lambda,\chi) = D_{2(n-2),\,2(n-2)}\,(P_{n-1}^2 \operatorname{sh}\Phi_{n-1} - P_{n-1}\,\lambda \operatorname{ch}\Phi_{n-1}) + \\ + D_{2(n-2)+1,\,2(n-2)}\,(P_{n-1} \operatorname{ch}\Phi_{n-1} - \lambda \operatorname{sh}\Phi_{n-1}) \qquad (17b)$$

Henceforth if function β has only one index then it denotes a function that is defined in every layer, viz:

$$\beta_k(\lambda,z) = \beta_{k,j}(\lambda,z) \qquad \text{if} \quad z_{j-1} > z > z_j$$

Let us see whether function $\beta_k(\lambda,z)$ that belongs to any root χ_k of equation $D(\lambda,\chi) = 0$ is generated by sine and cosine functions in at least one layer, viz. in at least one layer $w_j\,\chi_k > \lambda^2$. This is important because it ensures the existence of a smallest χ_k, viz. series (8) is actually an infinite sum in only one direction and the first term of the series will be determinant for late times. Assume that contrary to our statement $\beta_k(\lambda,z)$ is a linear combination of hyperbolic functions in every layer which means that Eqs. (15b), (16b) and (17b) are valid when determinant $D(\lambda,\chi)$ is computed. As $\Phi_1 < 0$, $D_{2,2} < 0$ and $D_{3,2} > 0$ follow from Eq. (15b). Taking $\Phi_{j+1} < 0$ into account it results from Eq. (16b) that this property is hereditary from layer to layer, viz. $D_{2j,\,2j} < 0$ and $D_{2j+1,\,2j} > 0$ if $0 < j < n-1$. Finally taking it into consideration that $\Phi_{n-1} < 0$, $D(\lambda,\chi) > 0$ is also true according to (17b), which means that equation $D(\lambda,\chi) = 0$ has no root. In consequence a function $\beta_k(\lambda,z)$ belonging to any χ_k root of the equation is a linear combination of sine and cosine functions in at least one layer.

The computation of kernel function $X(\lambda,z,t)$ has to be started with the determination of roots χ_k of equation $D(\lambda,\chi) = 0$, which is the most crucial part of the process. A numerical method is required that makes it unnecessary to compute determinant $D(\lambda,\chi)$ too many times in which case one of the advantages of the method, viz. the speed, could be lost. With the knowledge of χ_k we have to compute values $P_{k,j}$ in Eq. (13), then to solve the linear system of equations (14). Since function $\alpha_{k,j}(t)$ Eq. (11) contains a constant that will be determined later, constant $b_{k,1}$ can be chosen to equal 1 in the system of equations (14) and $a_{k,1}$ can be expressed by Eq. (14a). Knowing $a_{k,j-1}$ and $b_{k,j-1}$ Eqs. (14b) and (14c) make it possible to determine $a_{k,j}$ and $b_{k,j}$. Constants c_k can be determined by applying condition (7) after substituting formula (8) into (7) with t=0.

$$\sum_{k=1}^{\infty} c_k\,\beta_k(\lambda,z) = e^{\lambda z} \qquad (18)$$

Formally, Eq. (18) is the series expansion of function $e^{\lambda z}$ in term of functions $\beta_k(\lambda,z)$. Before determining constants c_k on this basis we have to prove the orthogonality of functions $\beta_k(\lambda,z)$. For a given layered model the definition of the scalar product defined in interval $(z_{n-1}, 0)$ in the space of continuously differentiable functions can be given as:

$$\langle f, g\rangle = \sum_{j=1}^{n-1} \int_{z_j}^{z_{j-1}} f(z)\, g(z)\, w_j \, dz \tag{19}$$

For the computation of the scalar product of functions $\beta_k(\lambda,z)$ and $\beta_l(\lambda,z)$ (Eq. 19) belonging to different roots χ_k and χ_1 of equation $D(\lambda,\chi) = 0$ let us take the integral that is valid for the j-th layer:

$$\int_{z_j}^{z_{j-1}} \beta_{k,j}(\lambda,z)\, \beta_{l,j}(\lambda,z)\, w_j \, dz =$$

$$= \frac{1}{\chi_k - \chi_j}\left(\left[-\frac{\partial}{\partial z}\beta_{k,j}\,\beta_{l,j}\right]_{z_j}^{z_{j-1}} + \left[\beta_{k,j}\frac{\partial}{\partial z}\beta_{l,j}\right]_{z_j}^{z_{j-1}}\right) \tag{20}$$

Integral (20) can be obtained by partial integration by considering differential equation (9) relating to function $\beta_{k,j}$. During the computation of scalar product (19), as integrals (20) corresponding to the layers are summed up, terms $\beta_k(\lambda,z)$ and $\frac{\partial}{\partial z}\beta_k(\lambda,z)$ that belong to inner layer boundaries will cancel out because of the continuity. However terms corresponding to the surface and to the lowest layer boundary will cancel out owing to boundary conditions (6). Thus the system of functions $\beta_k(\lambda,z)$ is orthogonal to the scalar product definied by formula (19). As $\beta_k(\lambda,z)$ is only orthogonal but not orthonormal, normalization is required to compute expansion coefficients c_k.

$$c_k = \frac{\langle e^{\lambda z}, \beta_k \rangle}{\langle \beta_k, \beta_k \rangle} \tag{21}$$

When we compute the numerator of fraction (21) only the term that corresponds to the surface will not cancel out when integrals are summed up.

$$< e^{\lambda z}, \beta_k > = \sum_{j=1}^{n-1} \int_{z_j}^{z_{j-1}} e^{\lambda z} \beta_{k,j}(\lambda,z)\, w_j \; dz = \frac{2\lambda a_{k,1}}{\chi_k} \tag{22}$$

Calculation of the denominator of (21) is a little more complicated but as in this case there are integrals of analytical functions and those can be computed by partial integration there is no need for numerical integration.

Tables I and *II* contain values of kernel function $X(\lambda,z,t)$ for different time values that were computed by the finite difference method and by formula (8).

time	$X(\lambda,z,t)$ finite difference method	$X(\lambda,z,t)$ series expansion
0.15020E-04	0.506120E-07	0.466440E-07
0.18399E-04	0.456334E-07	0.433343E-07
0.22154E-04	0.414997E-07	0.402167E-07
0.26659E-04	0.377440E-07	0.370948E-07
0.32391E-04	0.342071E-07	0.339257E-07
0.39050E-04	0.310262E-07	0.309217E-07
0.47310E-04	0.281279E-07	0.280964E-07
0.57072E-04	0.255988E-07	0.255910E-07
0.68712E-04	0.234067E-07	0.234051E-07
0.82980E-04	0.215164E-07	0.215161E-07
0.99876E-04	0.200088E-07	0.200087E-07
0.12015E-03	0.188494E-07	0.188494E-07
0.14456E-03	0.180045E-07	0.180046E-07
0.17384E-03	0.174167E-07	0.174168E-07
0.20914E-03	0.170042E-07	0.170043E-07
0.25156E-03	0.166867E-07	0.166871E-07
0.30000E-03	0.164108E-07	0.164113E-07

Table I. Computational results of kernel function $X(\lambda,z,t)$ by finite difference method (second column) and by formula (8) (third column). Parameters: $\rho_1=10$ Ωm, $\rho_2=100$ Ωm, $\rho_3=\infty$ Ωm, $d_1=50$ m, $d_2=50$ m, $\lambda = 0.001$

I. táblázat. Az $X(\lambda,z,t)$ magfüggvény számítása a véges differenciák módszerével (2. oszlop) és a (8) képlettel (3. oszlop). Paraméterek: $\rho_1=10$ Ωm, $\rho_2=100$ Ωm, $\rho_3=\infty$ Ωm, $d_1=50$ m, $d_2=50$ m, $\lambda = 0.001$

Табл. I. Расчет ядровой функции $X(\lambda,z,t)$ методом конечных разностей (столбец 2) и по формуле (8) (столбец 3). Параметры: $\rho_1=10$ омм, $\rho_2=10$ омм, $\rho_3=$ омм, $d_1=5$ м, $d_2=50$ м, $\lambda = 0.001$

It can be seen that for early times the results given by the two methods are different while for late times the difference between the two columns is less than 0.1 %. Computations were made by taking into account the first three terms of series expansion (8). The running time applying equation (8) is at least two orders less than in the case of the finite difference method. Taking into consideration that $\nu_{k,j} = w_j \chi_k$ and numbers χ_k form an ascending monotone series keeping to an infinite limit, from Eq. (11) it can be seen that for late times it is enough to compute only some of the first terms of series (8). Namely according to the effect of the exponential function further terms of the series are several orders less.

time	$X(\lambda,z,t)$	
	finite difference method	series expansion
0.15020E-04	0.133314E-07	0.122711E-07
0.18399E-04	0.111302E-07	0.104967E-07
0.22154E-04	0.929339E-08	0.893233E-08
0.26659E-04	0.766964E-08	0.748458E-08
0.32291E-04	0.623251E-08	0.615204E-08
0.39050E-04	0.505931E-08	0.502986E-08
0.47310E-04	0.411569E-08	0.410739E-08
0.57072E-04	0.339869E-08	0.339722E-08
0.68712E-04	0.285418E-08	0.285443E-08
0.82980E-04	0.243411E-08	0.243465E-08
0.99876E-04	0.212551E-08	0.212607E-08
0.12015E-03	0.190150E-08	0.190205E-08
0.14456E-03	0.174634E-08	0.174690E-08
0.17384E-03	0.164559E-08	0.164618E-08
0.20914E-03	0.158266E-08	0.158331E-08
0.25156E-03	0.154206E-08	0.154277E-08
0.30000E-03	0.151244E-08	0.151324E-08

Table II. Computational results of kernel function $X(\lambda,z,t)$ by finite difference method (second column) and by formula (8) (third column). Parameters: ρ_1=100 Ωm, ρ_2=10 Ωm, $\rho_3=\infty$ Ωm, d_1=50 m, d_2=50 m, λ = 0.001

II. táblázat. Az $X(\lambda,z,t)$ magfüggvény számítása a véges differenciák módszerével (2. oszlop) és a (8) képlettel (3. oszlop). Paraméterek: ρ_1=100 Ωm, ρ_2=10 Ωm, $\rho_3=\infty$ Ωm, d_1=50 m, d_2=50 m, λ = 0.001

Табл. II. Расчет ядровой функции $X(\lambda,z,t)$ методом конечных разностей(столбец 2) и по формуле (8) (столбец 3). Параметры: ρ_1=100 омм, ρ_2=10 омм, $\rho_3=\infty$ омм , d_1=50 м, d_2=50 м, λ = 0.001

4. Computing of the vertical component of the magnetic field strength at the midpoint of a circular induction loop

In consequence of the definition of vector potential (2) in the case of electric dipole:

$$H_z(t) = -\frac{\partial}{\partial z} A_x$$

Applying this to generate integral A_x of Eq. (4) we get the vertical component of the magnetic field strength:

$$H_z(t) = \frac{I\,dl\,y}{4\,\pi\,r} \int_0^\infty J_1(\lambda r)\,\lambda\,X(\lambda,z,t)\,\mathrm{d}\lambda$$

For a circular transmitter loop of radius r and with the receiver in the centre of the circle:

$$H_z(t) = \frac{I\,r}{2} \int_0^\infty J_1(\lambda r)\,\lambda\,X(\lambda,z,t)\,\mathrm{d}\lambda$$

Let us substitute its series form (Eq. 8) for kernel function $X(\lambda,z,t)$:

$$H_z(t) = \frac{I\,r}{2} \int_0^\infty J_1(\lambda r)\,\lambda \sum_{k=1}^{\infty} a_{k,1}\,c_k\,e^{-\frac{\chi_k}{\mu\,\sigma_1}t}\,\mathrm{d}\lambda \qquad (23)$$

As in practice the time derivative of the magnetic field strength is commonly measured let us derive equation (23) in terms of time and let us put the value given by formula (21) in the place of c_k:

$$\frac{\partial}{\partial t} H_z(t) = \frac{I\,r}{2\,\mu\,\sigma_1} \int_0^\infty J_1(\lambda r)\,\lambda \sum_{k=1}^{\infty} \frac{2\,P_{k,1}^2}{\lambda\,\langle\beta_k,\beta_k\rangle}\,e^{-\frac{\chi_k}{\mu\,\sigma_1}t}\,\mathrm{d}\lambda \qquad (24)$$

Thus transient curves measured by a central induction loop (CIL) array can be calculated by formula (24). We would mention that if we apply kernel function $X(\lambda,z,t)$, that can be obtained by formula (8), the transient field of a vertical magnetic field can be calculated, too. The integral that contains the Bessel function can be computed by filtering [ANDERSON 1979]. In *Table III.* we show the comparison of field values computed by different methods for a three-layered model. Henceforth we denote by n_s the number of terms that will be taken into account from series (8). The second column of the table contains $(\partial H_z(t)/\partial t)$ values computed by the

time	dH_z/dt	apparent resistivity	dH_z/dt	apparent resistivity
0.89000E-04	-0.22504E+01	66.086	-0.18666E+01	75.449
0.11204E-03	-0.16354E+01	55.917	-0.14925E+01	59.449
0.14106E-03	-0.11949E+01	47.120	-0.11521E+01	48.351
0.17758E-03	-0.86452E+00	39.976	-0.86540E+00	40.242
0.22356E-03	-0.61168E+00	34.441	-0.61191E+00	34.432
0.28144E-03	-0.41838E+00	30371	-0.41966E+00	30.306
0.35432E-03	-0.27511E+00	27.510	-0.27602E+00	27.447
0.44606E-03	-0.17535E+00	25.627	-0.17406E+00	25.573
0.56155E-03	-0.10498E+00	24.545	-0.10526E+00	24.499
0.70695E-03	-0.60949E-01	24.153	-0.61099E-01	24.113
0.89000E-03	-0.34005E-01	24.394	-0.34081E-01	24.357
0.11204E-02	-0.18264E-01	25.253	-0.18299E-01	25.220
0.14106E-02	-0.94625E-02	26.760	-0.94779E-02	26.730
0.17758E-02	-0.47401E-02	28.980	-0.47463E-02	28.955
0.22356E-02	-0.23014E-02	32.028	-0.23041E-02	32.002
0.28144E-02	-0.10859E-02	36.057	-0.10872E-02	36.027
0.35432E-02	-0.49945E-03	41.274	-0.50012E-03	41.237
0.44606E-02	-0.22459E-03	47.947	-0.22489E-03	47.904
0.56155E-02	-0.99026E-04	56.421	-0.99134E-04	56.379
0.70695E-02	-0.42929E-04	67.133	-0.42952E-04	67.110

Table III. Comparison of $(\partial H_z(t)/\partial t)$ values computed by the spectral technique (second column) and by formula (24) (fourth column). Layer parameters: n=3, ρ_1=100 Ωm, ρ_2 =10 Ωm, ρ_3=∞ Ωm, d_1=50 m, d_2=50 m

III. táblázat. Spektrál módszerrel (2. oszlop) és a (24) képlettel (4. oszlop) számított $(\partial H_z(t)/\partial t)$ értékek összehasonlítása. A rétegparaméterek: n=3, ρ_1=100 Ωm, ρ_2=10 Ωm, ρ_3=∞ Ωm, d_1=50 m, d_2=50 m

Табл. III. Сравнение значений $(\partial H_z(t)/\partial t)$, полученных спектральным способом (столбец 2) и по формуле (24) (столбец 4). Параметры: n =3, ρ_1=100 омм, ρ_2=10 омм, ρ_3=∞ омм, d_1=50 м, d_2=50 м

spectral method and in the fourth column values computed by formula (24) in case of n_s=1 are. At early times there is a little deviation but from the fourth time value it is less than 0.5 %. The speed of computation is one order greater than that of spectral technique. If n_s>1 then the accuracy improves even for early times, but it proportionally increases the running time. The third and fifth columns of Table III. contain apparent resistivity values near $(\partial/\partial t)\ H_z$ values. In *Fig. 1* a comparison can be made between results computed by n_s=1, 2 and 4 and by the spectral technique. For early times the apparent resistivity curve corresponding to n_s=1 is above the resistivity value of the first layer (30 Ωm). The curve of n_s=4 is almost equal to the curve computed by the spectral technique. In the time interval corresponding to the measuring range of the transient equipment the applicability and values n_s of the described method depend on the layer parameters. For thick and conductive layers only the greater values of n_s can give adequate results whereas in the case of thin and non-conductive layers even n_s=1 gives an accurate result. If the layer parameters are such that in the major part of the measuring interval even with n_s=4 we cannot get an acceptable result then it is only worth applying the spectral technique. *Fig. 2* shows what restriction it means that the described computation method works only in the case of a non-conductive basement. The computed apparent resistivity curves of a three-layered model are drawn on each other and the resistivity of the basement changes (500 Ωm, 1000 Ωm,

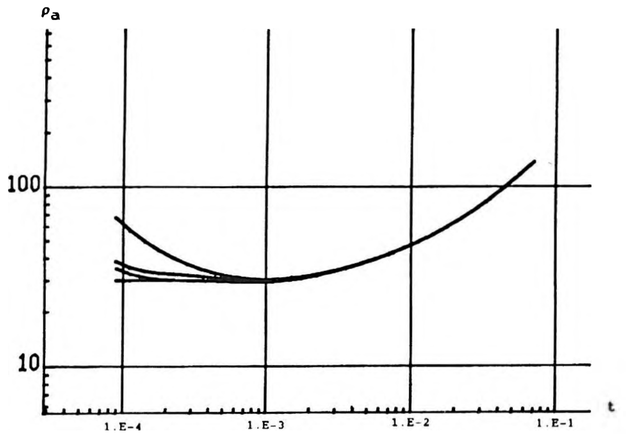

Fig. 1. Transient curves computed by spectral technique (lowest curve) and on the basis of formula (24). Layer parameters: n=3, ρ_1=30 Ωm, ρ_2=100 Ωm, ρ_3=∞ Ωm, d_1=200 m, d_2 =600 m, r=50 m. The curves from top to bottom were computed by taking into account terms 1, 2 and 4 of series (24)

1. ábra. Spektrál módszerrel (legalsó görbe) és a (24) képlet alapján számított tranziens görbék. A rétegparaméterek: n=3, ρ_1=30 Ωm, ρ_2=100 Ωm, ρ_3=∞ Ωm, d_1=200 m, d_2 =600 m, r =50 m. A különböző görbék felülről lefelé a (24) sor 1, 2, illetve 4 tagjának figyelembevételével készültek

Рис.1. Кривые переходного процесса, полученные спектральным способом и по формуле (24). Параметры слоев: n =3, ρ_1=30 ОММ, ρ_2=100 ОММ, ρ_3=∞ ОММ, d_1=200 М, d_2=600 М, r= 50 М. Разные кривые получены при учете 1, 2 и 4-ого члена формулы (24)

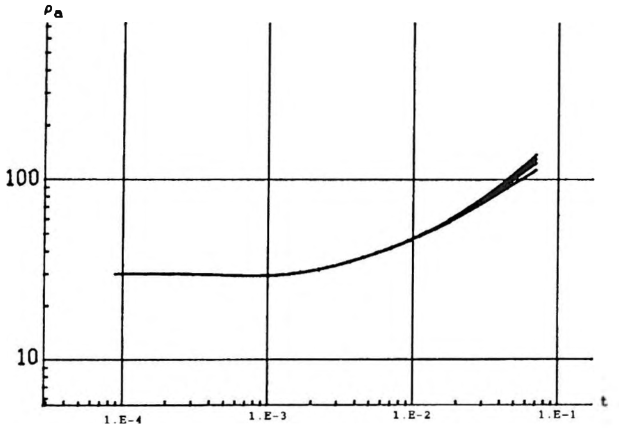

Fig. 2. Curves computed by spectral technique to demonstrate the effect of a non-conductive basement. Layer parameters: n=3, ρ_1=30 Ωm, ρ_2=100 Ωm, ρ_3=500, 1000, 2000 and ∞ Ωm, d_1=200 m, d_2=600m

2. ábra. Spektrál módszerrel számított tranziens görbék a rosszul vezető aljzat hatásának a szemléltetésére. A rétegparaméterek: n=3, ρ_1=30 Ωm, ρ_2=100 Ωm, ρ_3=500, 1000, 2000 és ∞ Ωm, d_1=200 m, d_2=600m

Рис.2. Кривые переходного процесса, расчитанные спектральным способом, для иллюстрации влияния плохопроводящего фундамента. Параметры слоев: n =3, ρ_1=30 омм, ρ_2=100 омм, ρ_3=500, 1000, 2000 и ∞ омм, d_1=200 м, d_2=600 м

2000 Ωm and non-conductive basement). For the given model deviations between the curves even for late times are small.

Conclusions

Formula (23) makes it possible to compute transient curves faster than till now, which essentially makes it quicker to interpret measured curves by curve fitting. Though the applicability of the method is restricted by assuming a non-conductive basement, and that for certain models it is inaccurate for early times, for most of the models that occur in practice it can be applied. One of the possible procedures of interpretation is to apply this computational method at the beginning, and when we only have to fit that part of the curve that belongs to early times we can change to the curve computation spectral technique.

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RÉTEGZETT FÉLTÉR FELSZÍNÉN KIALAKULÓ TRANZIENS ELEKTROMÁGNESES TÉR GYORS SZÁMÍTÁSA

PRÁCSER Ernő

Az időtartománybeli elektromágneses terek spektrál módszerrel, azaz a frekvenciatartománybeli értékekre alkalmazott inverz Fourier transzformálttal, vagy a Maxwell egyenletek időtartománybeli megoldásával számíthatók. Az utóbbi elven alapuló, a rétegzett féltér esetére érvényes és a teljes időtartományban pontos számítások általában időigényesek. Ennek az az oka, hogy a tranziens teret meghatározó képletek levezetésekor olyan parciális differenciálegyenlet is fellép, amelynek a megoldása nem állítható elő analitikus függvények segítségével. A parciális differenciálegyenletek numerikus megoldása viszont rendkívül időigényes. Ezzel szemben a spektrál módszerrel történő számítások során a frekvenciatartományban felmerülő differenciálegyenlet megoldása kifejezhető analitikus függvényekkel. Abban az esetben azonban, amikor nem törekszünk a teljes időtartományban pontos megoldásra, az időtartományban fellépő parciális differenciálegyenlet megoldása is előállítható analitikus függvények segítségével. Egy ilyen esetet ismertet a cikk, amely szigetelő aljzat esetére érvényes és a késői időkre pontos, aszimptotikus megoldáson alapul.

БЫСТРОЕ ВЫЧИСЛЕНИЕ ЭЛЕКТРОМАГНИТНОГО ПОЛЯ ПЕРЕХОДНОГО ПРОЦЕССА НА ПОВЕРХНОСТИ СЛОИСТОГО РАЗРЕЗА

Эрнё ПРАЧЕР

Электромагнитные поля во временной области могут быть рассчитаны спектральным способом, — то есть при помощи обратного преобразования Фурье, примененного для величин заданных в частотной области, или путем решения уравнений Максвелла. Способы расчетов, основанные на последнем принципе, применяемые для слоистого полупространства и точные во всем диапазоне времени, как правило, требуют много машинного времени. Это связано с тем, что при выведении формул, определяющих поле переходного процесса, имеется и такое частное дифференциальное уравнение, решение которого нельзя найти с помощью аналитических функций, а числовое решение таких уравнений является трудоемким. Наоборот, дифференциальные уравнения, возникающие при решении задачи в частотной области, можно выражать аналитическими функциями. Если не требуется точное решение во всей временной области, то и дифференциальные уравнения, заданные во временной области, могут решаться при помощи аналитических функций. В статье излагается такой случай, когда при наличии фундамента-изолятора для поздних времен получим точное решение асимптотическим способом.