GEOPHYSICAL TRANSACTIONS 1987
Vol. 33. No. 2. pp. 113–122

ASPECTS OF FINITE DIFFERENCE MODELLING OF THE ELECTROMAGNETIC FIELD OF AN OSCILLATING ELECTRIC DIPOLE

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Assuming a harmonic time dependent electric source parallel to the strike the paper deals with the determination of its electromagnetic field for the plane which is perpendicular to the strike and contains the source. The chosen numerical procedure is the finite difference method. A direct method taking the blocked-tridiagonal structure of coefficient matrix into consideration is recommended for solving the resulting linear system. After solving the set of equations for numerous spatial wavenumbers the field components are determined numerically by inverse Fourier transformation. The number and distribution of discrete spatial wavenumbers need to be planned. The way of planning is shown by an example.

Keywords: 2-D structure, conductivity, electric dipole source, finite difference method, Fourier transformation

1. Introduction

If we assume a 2-D conductivity structure the determination of the electromagnetic field of an electric source treated as a point source is a 3-D problem. Fourier transformation may be used to substitute the three-dimensional problem for a series of two-dimensional problems. In the case of direct current sounding Laplace's and Poisson's equations and in the present case (frequency sounding) Maxwell's equations are to be Fourier transformed over the strike direction. As a result of Fourier transformation the series of 2-D problems belongs to different spatial wavenumbers; inverse Fourier transformation enables the field to be calculated. Dey and Morrison [1979] developed this method to solve the three-dimensional potential distribution about a point source of direct current located in or on the surface of a half-space containing 2-D conductivity structure. Stoyer and Greenfield [1976] worked out the response of a 2-D earth to an oscillating magnetic dipole source in this way. However, their general formulation can be applied to electric dipole sources.

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Paper presented at the 31st International Geophysical Symposium, Gdansk, 30 September–3 October, 1986

2. Mathematical formulation

The basic relationships governing this electromagnetic phenomenon are Maxwell's equations. Assuming only $e^{j\omega t}$ time dependent electric source they are:

$$\text{rot } \vec{E} = -j\omega\mu\vec{H} \quad (1)$$
$$\text{rot } \vec{H} = (\sigma + j\omega\varepsilon)\vec{E} + \vec{i}_s \quad (2)$$

where $\vec{i}_s$ is the current density. If equations (1), (2) are reduced to components in the x, y, z directions the Fourier transforms of the equations can be determined over the strike direction (x). If G denotes any component of $\vec{E}$ or $\vec{H}$ the Fourier transform of G over x is:

$$\tilde{G}(k_x, y, z) = \int_{-\infty}^{\infty} G(x, y, z)e^{-jk_x x}\,dx \quad (3)$$

The Fourier transform of the function $\frac{\partial G}{\partial x}$ over x is $(-jk_x)$ times $\tilde{G}(k_x, y, z)$ because $G(x, y, z)$ vanishes as $x \to \pm\infty$ if the source is placed to the origin of the Cartesian system of coordinates. Taking into consideration the abovementioned relationships and assuming only an electric source parallel to the strike the densest form of the Fourier transform of the component equations (1), (2) is:

$$-\frac{\partial}{\partial y}\left(\frac{1}{\mathfrak{z}^M}\frac{\partial \tilde{H}_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{1}{\mathfrak{z}^M}\frac{\partial \tilde{H}_x}{\partial z}\right) - jk_x\frac{\partial \xi}{\partial y}\frac{\partial \tilde{E}_x}{\partial z} + jk_x\frac{\partial \xi}{\partial z}\frac{\partial \tilde{E}_x}{\partial y} + \mathfrak{y}^M\tilde{H}_x = 0 \quad (4)$$

$$-\frac{\partial}{\partial y}\left(\frac{1}{\mathfrak{z}^E}\frac{\partial \tilde{E}_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{1}{\mathfrak{z}^E}\frac{\partial \tilde{E}_x}{\partial z}\right) + jk_x\frac{\partial \xi}{\partial y}\frac{\partial \tilde{H}_x}{\partial z} - jk_x\frac{\partial \xi}{\partial z}\frac{\partial \tilde{H}_x}{\partial y} + \mathfrak{y}^E\tilde{E}_x = -\tilde{i}_{sx} \quad (5)$$

Equations (4), (5) are called the Transverse Magnetic *(TM)* and the Transverse Electric *(TE)* equation, respectively. In these equations $\tilde{H}_x$, $\tilde{E}_x$ are the Fourier transforms of H_x and E_x over x; $\tilde{i}_{sx}$ is the Fourier transform of the electric source term in the strike direction. If k denotes the wavenumber, ξ, *TM* admittance, *TE* impedance, *TE* admittance, *TM* impedance can be defined after STOYER [1974] in the following way: $\xi = (k_x^2 - k^2)^{-1}$; $\mathfrak{y}^M = j\omega\mu$; $\mathfrak{z}^E = (1 - k_x^2/k^2)\mathfrak{y}^M$; $\mathfrak{y}^E = (\sigma + j\omega\varepsilon)$; $\mathfrak{z}^M = (1 - k_x^2/k^2)\mathfrak{y}^E$. These parameters are constant within each grid element *(Fig. 1)*.

Without solving the coupled partial differential equations (4), (5) it is easy to see that $\tilde{E}_x(k_x, y, z)$ must be even in k_x and $\tilde{H}_x(k_x, y, z)$ must be odd in k_x. To verify the assertion above observe that $\tilde{i}_{sx}$ is an even function in k_x on the right side of equation (5). Therefore the superposition of terms occurring on the right side of equation (5) must be even in k_x as well. It follows from this that each term on the right side of equation (5) must be even in k_x. It is possible if and only if function $\tilde{E}_x(k_x, y, z)$ is even and function $\tilde{H}_x(k_x, y, z)$ is odd in k_x. In order to get the solution in the space domain the solution of equations (4),

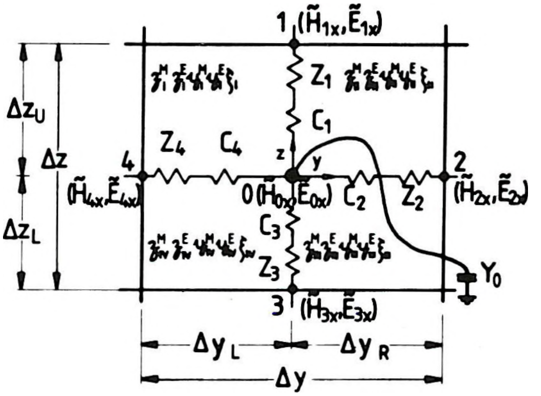

Fig. 1. Grid element of two-dimensional conductivity model in the (k_x, y, z) domain

1. ábra. Kétdimenziós vezetőképesség modell elektromos szimulációjának részlete a (k_x, y, z) tartományban

Рис. 1. Фрагмент электрического боспроизведения двумерной модели электропроводности области (k_x, y, z).

(5) is to be inverse Fourier transformed. This transformation for the plane which is perpendicular to the strike and contains the source becomes simple because $x=0$. This fact further simplifies the inverse Fourier transformations:

$$E_x(0, y, z) = \frac{1}{\pi} \int_0^\infty \tilde{E}_x(k_x, y, z)\, \mathrm{d}k_x \tag{6}$$

$$H_x(0, y, z) = 0 \tag{7}$$

For a source-free area $\tilde{H}_y$, $\tilde{H}_z$, $\tilde{E}_y$, $\tilde{E}_z$ can be expressed as a function of $\tilde{E}_x$ and $\tilde{H}_x$. These are used for calculating the other components of the field:

$$H_y(0, y, z) = \frac{1}{\pi} \int_0^\infty \left(\frac{-\dfrac{\partial \tilde{E}_x}{\partial z}}{(1 - k_x^2/k^2)\mathrm{j}\omega\mu} - \frac{\mathrm{j}k_x}{(k_x^2 - k^2)} \frac{\partial \tilde{H}_x}{\partial y} \right) \mathrm{d}k_x \tag{8}$$

$$H_z(0, y, z) = \frac{1}{\pi} \int_0^\infty \left(\frac{\dfrac{\partial \tilde{E}_x}{\partial y}}{(1 - k_x^2/k^2)\mathrm{j}\omega\mu} - \frac{\mathrm{j}k_x}{(k_x^2 - k^2)} \frac{\partial \tilde{H}_x}{\partial z} \right) \mathrm{d}k_x \tag{9}$$

$$E_y(0, y, z) = 0 \tag{10}$$

$$E_z(0, y, z) = 0 \tag{11}$$

3. Numerical process

As has already been pointed out the numerical procedure utilized is the finite difference method. The coupled partial differential equations have been given in (4), (5). The next step is to discretize them over a rectangular net.

3.1 Structure of the system of equations

Using a three-point difference operator for the second derivative and averaging each term on the left-hand side over the four quarter blocks surrounding a central node, the final form of the finite difference equations is:

$$\sum_{i=1}^{4} p_i \left[\frac{\tilde{H}_{ix} - \tilde{H}_{ox}}{-Z_i^M} + \frac{\tilde{E}_{ix} - \tilde{E}_{ox}}{C_i} \right] + Y_o^M \tilde{H}_{ox} = 0 \tag{12}$$

$$\sum_{i=1}^{4} p_i \left[\frac{\tilde{E}_{ix} - \tilde{E}_{ox}}{-Z_i^E} - \frac{\tilde{H}_{ix} - \tilde{H}_{ox}}{C_i} \right] + Y_o^E \tilde{E}_{ox} = -\tilde{i}_{sox} \Delta y \Delta z \tag{13}$$

where Z denotes lumped impedances, C denotes coupling terms between the central and one of the four neighbouring nodes, and Y denotes lumped admittances (Fig. 1). Similarly to 2-D magnetotellurics the lumped impedances are the parallel combinations of the impedances of two adjacent elements, and the lumped admittances are the parallel combinations of the admittances of the four elements surrounding a node.

The reciprocal values of coupling terms – which do not occur in magnetotellurics – are directly proportional to values ξ of the adjacent elements and are independent of the size of elements. For example Z_1^M, C_1 and Y_o^M can be given by the formulae:

$$\frac{1}{Z_1^M} = \frac{2}{\Delta z_U} \left(\frac{\Delta y_R}{\mathfrak{z}_{II}^M} + \frac{\Delta y_L}{\mathfrak{z}_I^M} \right) \tag{14}$$

$$\frac{1}{C_1} = 2\,\mathrm{j} k_x (\xi_I - \xi_{II}) \tag{15}$$

$$Y_0^M = \Delta y_L \Delta z_U \mathfrak{y}_I^M + \Delta y_R \Delta z_U \mathfrak{y}_{II}^M + \Delta y_R \Delta z_L \mathfrak{y}_{III}^M + \Delta y_L \Delta z_L \mathfrak{y}_{IV}^M \tag{16}$$

Using a Neumann-type boundary condition in equations (12) and (13), $p_i = 0$ if point P_i is outside the grid, and $p_i = 1$ in any other case. If we use a terminal-impedance type boundary condition, i.e. the edge of the mesh is grounded, the outward-directed lumped impedances are a fraction of the inward-directed lumped impedances. This can be accomplished by a suitable choice of p_i. In equation (13) $\tilde{i}_{sox}$ is the Fourier transform of the strike directional electric source term in finite difference form. The sources can be treated like distributed parameters [Stoyer 1974]. Oscillating electric dipoles are treated as point sources in the strike direction. If their lengths were finite in the strike direction, the Fourier transforms of sources would be oscillating functions in k_x. The strike directional source term on the surface is treated as a Heaviside function in the two adjacent quarter elements situated below the surface *(Fig. 2)*. The treatment above makes it possible to determine the Fourier transform of the electric source placed in the strike direction:

$$\tilde{i}_{sox} \approx \frac{1}{\frac{\Delta y_L}{2} \cdot \frac{\Delta z_L}{2}} \int_{z_0 - \frac{\Delta z_L}{2}}^{z_0} \int_{y_0 - \frac{\Delta y_L}{2}}^{y_0 + \frac{\Delta y_R}{2}} \tilde{i}_{sx} \, \mathrm{d}y \, \mathrm{d}z = \int_{-\infty}^{\infty} i_{sx} \delta(x) \, \mathrm{e}^{-\mathrm{j}k_x x} \, \mathrm{d}x = i_{sx} \qquad (17)$$

It means that there is only one node that has a source term different from zero, and it is equal to the applied electric source.

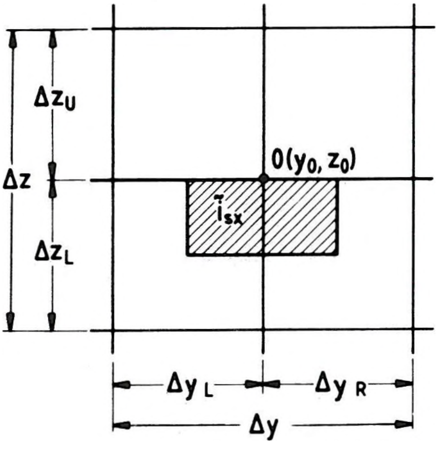

Fig. 2. Treatment of an electric source in strike direction on the surface

2. ábra. Felszínen levő, csapásirányú elektromos dipólus egyszerűsített vázlaza

Рис. 2. Упрощенная схема электрического диполя, параллельного простиранию и находящегося на поверхности.

After decomposition of finite difference equations (12), (13) to equations containing either real (R) or imaginary (I) terms we get four (TM_R, TM_I, TE_R, TE_I) equations belonging to a node. Progressing column-wise on the grid nodes from left to right and writing the equations successively the resulting linear set of equations has the form *(Fig. 3)*

$$[Q]\,\vec{X} = \vec{S} \qquad (18)$$

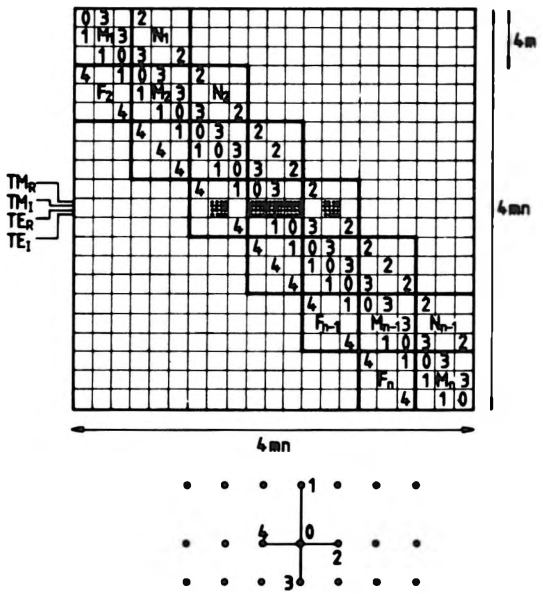

Fig. 3. Structure of a coefficient matrix belonging to a grid with three rows and seven columns

3. ábra. Három sorból és hét oszlopból álló rácshoz tartozó együtthatómátrix szerkezete

Рис. 3. Структура матричного коэффициента, принадлежащего сетке, состоящей из трех строчек и семи граф.

Coefficient matrix $[Q]$ has a band structure, it is a square and non-symmetrical matrix; $\vec{X}$ is the column vector of unknowns containing the components in the order of $\tilde{H}_x^R$, $\tilde{H}_x^I$, $\tilde{E}_x^R$, $\tilde{E}_x^I$; $\vec{S}$ is a column vector containing the Fourier transforms of source terms in finite difference form belonging to TM_R, TM_I, TE_R, TE_I equations. If we use a Neumann-type or terminal-impedance type boundary condition, $[Q]$ has a blocked-tridiagonal structure as well. If the grid has m rows and n columns and m is smaller than n it is worth numbering the nodes column-wise, because the size of blocks and $[Q]$ can be partitioned in the form:

$$[Q] = \{[F]_l[M]_l[N]_l\}_1^n = [L][U] \qquad (19)$$

where l denotes the number of the actual column, $[F]_l$ and $[N]_l$ are co-diagonal blocks, and $[M]_l$ represents main diagonal blocks. Each block has $4m.4m$ elements. Vectors $\vec{X}$ and $\vec{S}$ can be partitioned in the same manner: $\vec{X}_l$ and $\vec{S}_l$ belong to column l.

3.2 Solution of the set of equations

SCHECHTER [1960] suggested a reduction of Q to the form shown in equation (19) where L and U are square matrices, partitioned in the same manner as $[Q]$, of the form

$$[L] = \{[C]_l [I]_l [0]\}_1^n \tag{20}$$

$$[U] = \{[0] [A]_l [N]_l\}_1^n \tag{21}$$

where $[I]_l$ denotes unit matrix.

From equations (19), (20), (21) $[A]_l$ may be determined recursively:

$$[A]_1 = [M]_1 \tag{22}$$

$$[A]_l = [M]_l - [F]_l [A]_{l-1}^{-1} [N]_{l-1} \qquad l = 2, 3, \dots n. \tag{23}$$

Introducing $[U]\vec{X} = \vec{V}$ the first step is to solve $[L]\vec{V} = \vec{S}$. Its solution may be obtained recursively too:

$$\vec{V}_1 = \vec{S}_1 \tag{24}$$

$$\vec{V}_l = \vec{S}_l - [F]_l [A]_{l-1}^{-1} \vec{V}_{l-1} \qquad l = 2, 3, \dots, n. \tag{25}$$

The solution of $[U]\vec{X} = \vec{V}$ is equal to those of (18):

$$\vec{X}_n = [A]_n^{-1} \vec{V}_n \tag{26}$$

$$\vec{X}_l = [A]_l^{-1} (\vec{V}_l - [N]_l \vec{X}_{l+1}) \qquad l = n-1, n-2, \dots, 2, 1 \tag{27}$$

Using the same geoelectric section and frequency with the same grid size but having another source position or another type of source ($\vec{R}$) one has only to change in the source terms of equations (24), (25):

$$\vec{V}_1 = \vec{R}_1 \tag{28}$$

$$\vec{V}_l = \vec{R}_l - [F]_l [A]_{l-1}^{-1} \vec{V}_{l-1} \qquad l = 2, 3, \dots, n. \tag{29}$$

The advantage of this decomposition is obvious from the above: it reduces the $4mn.4mn$ matrix ($[Q]$) inversion to n $4m.4m$ matrix ($[A]_l$) inversions.

This direct method was applied to solve that part of the problem related to the homogeneous half-space. We were restricted to the case of $k_x = 0$. The frequency 20 kHz, the grid 12 by 25 nodes, and the conductivity of the earth was 0.01 mho/meter. The number of unknowns is 1200. The distribution of the absolute value of strike directional electric field in Vm^{-1} and the structure of the grid are illustrated in *Figure 4*. The source was placed on the surface (6th row) and into the 13th column, its current intensity was 10 A. The smallest grid elements, having a length of 5m and a width of 5m, are next to the source.

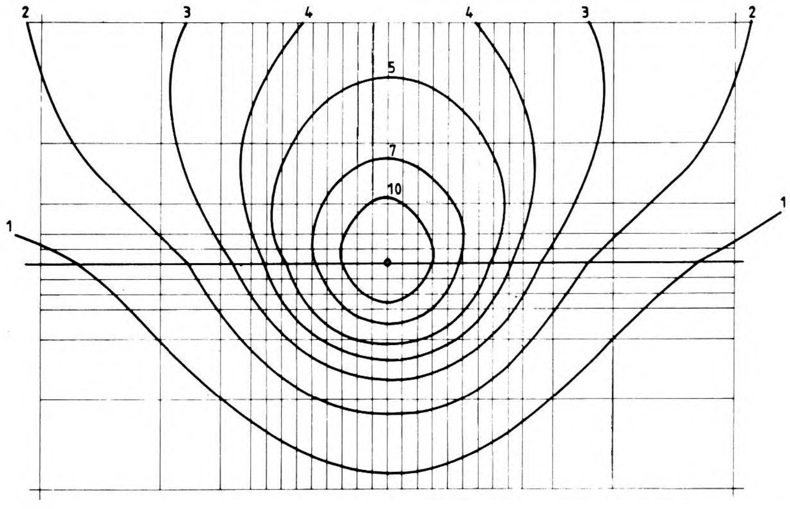

Fig. 4. Distribution of Fourier transformed strike directional field component in V/m

4. ábra. Csapásirányú elektromos térkomponens Fourier transzformáltjának izovonalas térképe V/m-ben

Рис. 4. Карта изолиний V/m трансформата Фурье электрической компоненты поля, параллельной простиранию.

3.3 Planning of wavenumber domain

The set of equations (18) has to be solved for different k_x values. The simultaneous aim is to compute for as few k_x values as possible and to achieve suitable accuracy. For this reason the number and the distribution of discrete spatial wavenumbers need to be planned. They depend upon the geoelectric and geometric parameters of structures, the frequency and the transmitter–receiver arrangement. We can only plan for 1—D geoelectric sections. The electromagnetic field of an electric dipole source placed on a homogeneous or a layered half-space can be computed for a line which is parallel to the strike direction and is situated on the surface. After this these functions are to be Fourier transformed over x. In order to get accurate space domain values by inverse Fourier transformation for $x=0$, we have to choose the appropriate number and distribution of k_x values. *Figure 5* shows the Fourier transforms of strike directional electric field components computed numerically from the space domain values over half-spaces of different conductivity [Takács 1983]. Assuming a horizontal electric dipole source in the x direction on the surface, the electric field has been determined for the line which is parallel to the x-direction and is situated 900 metres from the source on the surface. The current intensity of the source is 1 A. Formulae derived by Bannister and Dube [1978] allow ready computation of these functions. The Fourier transform of component E_x

with indices 1, 2, 3 refers to $\varrho_1 = 190\ \Omega m$, $f_1 = 800$ Hz; $\varrho_2 = 24\ \Omega m$, $f_2 = 244$ Hz; $\varrho_3 = 60\ \Omega m$, $f_3 = 25$ Hz, respectively. Subsequent inverse Fourier transformation showed that the 13 k_x values selected logarithmically in the range of $0 < k_x \leqq$ $\leqq 1.648 \cdot 10^{-3}\ m^{-1}$ were sufficient to achieve an accuracy within 1% in the $x = 0$ plane. In this way, if we have a 2-D inhomogeneity embedded in a 1-D structure, we can determine approximately for which k_x values the set of equations is to be solved.

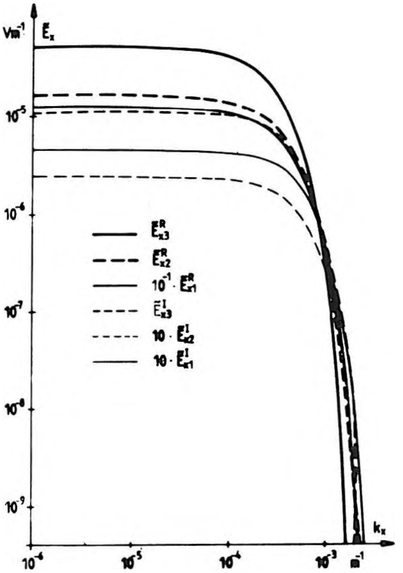

Fig. 5. Fourier transforms of strike directional electric field components over a half-space computed for a line parallel to the source in the strike direction

5. ábra. Csapásirányú mos dipólus csapásirányú mos tér komponenseinek Fourier transzformáltjai forrással párhuzamos felszíni vonal mentén homogén féltér esetén

Рис. 5. Трансформаты Фурье электрической компоненты поля, параллельной простиранию, над однородным полупространством, рассчитанные вдоль линии, параллельной источнику вдоль простирания.

4. Conclusions

Earlier the response of a 2-D earth to an oscillating magnetic dipole source was investigated. The present work describes a numerical method which makes it possible to determine the electromagnetic field of an oscillating electric dipole source placed on a 2-D earth. The electric source is parallel to the strike and it is treated as a point source. The result of a parity test is taken into consideration to determine the field components for the plane which is perpendicular to the strike and contains the source. The algorithm worked out by SCHECHTER has been applied to find the solution to the set of equations. On the basis of the numerical calculations the spatial wavenumber domains to be considered are equal under the condition of constant transmitter–receiver distance.

Acknowledgements

The author is indebted to Professor E. Takács who made it possible for him to work on this theme. Computation work on a CDC 3300 was financially supported by the Geophysical Department of the Technical University for Heavy Industry, Miskolc.

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MEGJEGYZÉSEK AZ OSZCILLÁLÓ ELEKTROMOS DIPÓLUS FORRÁS ELEKTROMÁGNESES TERÉNEK NUMERIKUS MODELLEZÉSÉHEZ

PETHŐ Gábor

Harmonikus, időtől függő, csapásiránnyal párhuzamos elektromos forrást feltételezve a cikk a csapásirányra merőleges, a forrást tartalmazó sík elektromágneses terének meghatározásával foglalkozik. Az alkalmazott numerikus eljárás a véges differenciák módszere. Az együttható mátrix blokktridiagonális szerkezetét figyelembe vevő direkt módszert javasolja a kapott lineáris egyenletrendszer megoldására. Számos térbeli hullámszámra megoldva az egyenletrendszert, a térkomponenseket inverz Fourier transzformáció segítségével határozza meg numerikusan. A diszkrét térbeli hullámszámok számát és eloszlását meg kell tervezni. Ezt az eljárást egy példán keresztül mutatja be.

ЦИФРОВОЕ МОДЕЛИРОВАНИЕ ЭЛЕКТРОМАГНИТНОГО ПОЛЯ ДВУМЕРНОГО КАЧАЮЩЕГОСЯ ЭЛЕКТРИЧЕСКОГО ДИПОЛЯ

Габор ПЕТЭ

В статье рассматривается определение электромагнитного поля плоскости, перпендикулярной простиранию и проходящей через источник, предполагая гармонический электрический источник, параллельный структурному простиранию и зависящий от времени. Применяется цифровой метод — способ конечных разностей. Для решения полученной линейной системы рекомендуется прямой метод, учитывающий блоковую, тридиагональную структуру матричного коэффициента. Вслед за решением системы уравнений нескольких объемных волновых чисел полевые компоненты определяются в цифровом виде, с помощью обратных трансформатов Фурье. Необходимо запланировать распределение и количество дискретных объемных волновых чисел, что иллюстрируется конкретным примером.