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RECURSION BAND-FILTERS AND THEIR DESIGN

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One phase of data processing in numerous geophysical (seismic, gravimetric, geoelectric) methods is the band-filtering of data. Filtering is frequently performed with the convolution of the input data system and another data system, the weight function of the filter. The filtered data are, depending on the length of the weight function given, a result of a certain number of multiplications and additions. The length of the weight function, and thus the number of operations according to output points is the higher, the narrower the frequency band to be passed or filtered out. In such cases, the application of recursion techniques is very advantageous. Its essence is that it uses, for the production of one output point, not only the input data, but also output data, filtered already earlier too. In this way a filtering effect identical with that of convolution filters is attained with much less operations. The general formula of the recursion algorithm is

$$y_n = \sum_{j=-N}^{M} a_j x_{n-j} - \sum_{i=1}^{L} b_i y_{n-i}, \tag{1}$$

where

y — the filtered output
a — the filter affecting the input
b — the filter affecting the earlier already filtered input
x — the input

It is assumed that the sampling interval is of unit value.

Formula (1) uses $M+N+1$ input points and L output points for the production of a single output value. Let us consider now the case $N=O$, $M=L$ and let us compute the transfer function of the operation. For this purpose (1) has to be written in the frequency domain:

$$F(\{y\}_m) = F(\{a_j\}_{j=0}^{M}) F(\{x\}_l - F(\{b_i\}_{i=1}^{M}) F(\{y\}_m) \tag{2}$$

where F indicates the Fourier-transform, and l runs over input points, m runs through every output points. After rearrangement:

$$F(\{y\}_m) = \frac{F(\{a_j\}_{j=0}^{M})}{1+F(\{b_i\}_{i=1}^{M})} F(\{x\}_l). \tag{3}$$

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The transfer function of the operation:

$$F(w)=\frac{F(\{a_j\}_{j=0}^{M})}{1+F(\{b_i\}_{i=1}^{M})}=\frac{a_0+ae^{-jW}+\ldots+a_Me^{-jWM}}{1+b_1e^{-jW}+\ldots+b_Me^{-jWM}}. \quad (4)$$

Since the sampling interval is of unit value, $-180° \le W \le 180°$, where $-180°$, $+180°$ is the Nyquist-interval.

If $z=e^{-jW}$ substituted in (4), it is represented in the so-called z-domain:

$$F(z)=\frac{a_0+az+\ldots+a_Mz^M}{1+b_1z+\ldots+b_Mz^M}=\frac{A_M(z)}{B_M(z)}. \quad (5)$$

$F(z)$ is the ratio of real-coefficient, complex variable, M-th degree polynomials. It is simply visible that the Nyquist-interval is represented on the unit circle of the z-plane, since $|z|=1$ and, if W varies from $-180°$ to $+180°$, z runs around the unit circle. The $F(z_0)$ value of the transfer function can be ordered to every z_0 point of the unit circle. If $F(z)$ is required to have a definite form, the polynomials $A_M(z)$ and $B_M(z)$ have to be chosen suitably. This can be attained by a proper position of the roots of the polynomials. It must be noted that if a z_i value is the root of any of the polynomials, then also $\bar{z}_i$ must be one. Namely, the coefficients of the polynomials agree with the coefficients of the filter; the latter must be, however, real.

In geophysical literature, the procedure was used for the first time by Shanks (1967), then by Mooney (1968) who discussed the design of notch filters, band-cut filters and band-pass filters in detail. As a simple example, a notch filter will be presented here. In the following, the roots of $A_M(z)$ will be called zero places, the roots of $B_M(z)$, on the other hand, poles. Be the zero places $z_0=e^{-j36°}$ resp. $\bar{z}_0=e^{+j36°}$, and the poles $z_0=1{,}01.e^{-j36°}$, resp. $\bar{z}_0=1{,}01.e^{+j36°}$. The amplitude characteristics and the position of the zero places, resp. poles are visible in Fig. 1. R_p indicates the distance of the pole from the origo. The sharpness of the characteristics can be varied. If R_p increases, the steepness decreases. This is shown in Fig. 1, where

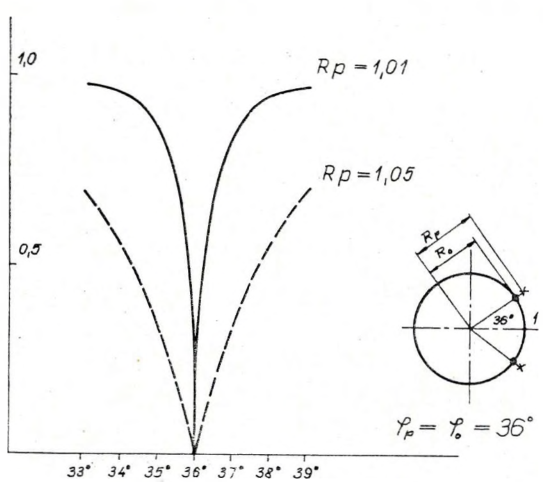

Fig. 1 Amplitude characteristics of notch filters R_p distances 1,05, resp. 1,01

1. ábra. Lyukszűrők amplitúdó-karakterisztikái. A választott R_p távolság 1,05 illetve 1,01

Рис. 1. Амплитудные характеристики фильтров-пробок при расстояниях R_p равных 1,05 и 1,01 соответственно notch filters of parameters $R_p = 1{,}05$ and $R_p = 1{,}01$ are presented. The transfer functions are

$$F(z) = \frac{(z - e^{-j36°})(z - e^{+j36°})}{(z - R_p e^{-j36°})(z - R_p e^{+j36°})} \tag{6}$$

where R_p is 1,01 or 1,05.

The filter has 5 points:

$$F(z) = \frac{\frac{1}{R_p^2} - \frac{2\cos 36°}{R_p^2} z + \frac{1}{R_p^2} z^2}{1 - \frac{2\cos 36°}{R_p} z + \frac{1}{R_p^2} z^2} \tag{7}$$

the filter points being:

$$a_0 = \frac{1}{R_p^2} \qquad a_1 = \frac{-2\cos 36°}{R_p^2} \qquad a_2 = \frac{1}{R_p^2}$$

resp.

$$b_1 = \frac{-2\cos 36°}{R_p} \qquad b_2 = \frac{1}{R_p^2}.$$

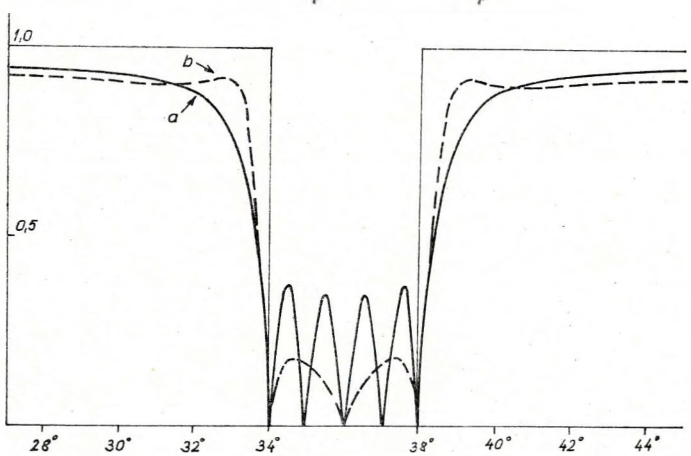

Fig. 2 Amplitude characteristics of band-reject filters; width of filtered-out band: 4° a) Zero/pole pairs established in 1° intervals; $R_p = 1{,}01$; b) Zero/pole pairs established in 2° intervals; pole places situated at $R_p = 1{,}008$; 1,05; 1,008 and $\varphi_p = 33{,}5°$; 36°; 38,5°

2. ábra. Sávvágó szűrők amplitúdókarakterisztikái. A kiszűrt sáv 4° szélességű. *a)* A zérus–pólus párokat 1°-ként helyeztük el. $R_p = 1{,}01$. *b)* A zérus–pólus párokat 2°-ként, a pólushelyeket $R_p = 1{,}008$; 1,05; 1.008 és $\varphi_p = 33{,}5°$; 36°; 38,5° értékeknek megfelelően helyeztük el

Рис. 2. Амплитудные характеристики полосно-заграждающих фильтров. Ширина заграждения полосы —4°. а) Пары нулевых полюсов размешены через 1°. Расстояние $R_p = 1{,}01$ б) Пары нулевых полюсов размешены через 2°, места полюсов выбраны в соответствии с величинами $R_p = 1{,}008$; 1,05; 1,008 и $\varphi_p = 35{,}5°$; 36°; 38,5°

If the filtering of a broader band is wanted, more zero-pole pairs must be placed besides each other. The slope depends also further on the R_p distances; in the exclusion range the amplitude characteristics will be better, the more densely are the zero-pole pairs spaced. This involves, of course, an increase in the number of the filter points. Mooney suggests 0,5°, resp. 1° zero place distances with identical R_p distances. In Fig. 2, the curve denoted by a is the amplitude characteristics of a filter cutting a 4° band. The centre of the band lies at 36°. The zero-pole pairs have been established at each 1°; R_p was 1,01 selected out of the parameters suggested by Mooney. The slope is very steep, but the elimination within the exclusion band is not sufficiently good. This can be improved, if the notch filter placed in the centre of the band is chosen for less steep, and the slope is gradually increased by a suitable selection of the R_p distances. Naturally this will deteriorate the steepness of the filter. This is clearly visible in Fig. 3, where the filtering-out of the afore-mentioned 4° band is our aim, the zero/pole pairs were situated at each 1°; R_p distances were 1,01; 1,025, 1,04; 1,025, 1,01. The zero places in this case, as always, were placed on unit circles.

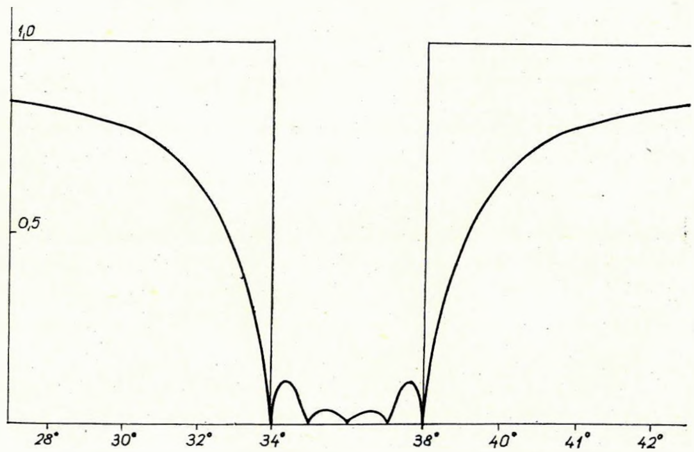

Fig. 3 Amplitude characteristics of a band-reject filter. Width of filtered-out band: 4°; zero/pole pairs in intervals of 2°; R_p values: 0,01; 1,025; 1,04; 1,025; 1,01

3. ábra. Sávvágó szűrő amplitúdókarakterisztikája. A kiszűrt sáv szélessége 4°. A zérus–pólus párokat 2°-ként helyeztük el. R_p= 1,01; 1,025; 1,04; 1,025; 1,01

Рис. 3. Амплитудная характеристика полосно-заграждающего фильтра. Ширина полосы заграждения — 4°. Пары нулевых полюсов размещены через 2°. Расстояния R_p выбирались равными 1,01; 1,025; 1,04; 1,025; 1,01

Another possibility for the improvement of the characteristics would be the densifying of zero, resp. pole places, e.g. to distances of 1/2°. This would, however, as mentioned, increase the number of filter-coefficients. In case of 1° spacing, the filter would consist of 19 points; with 1/2° spacing, of 35 points. This would be, of course, still advantageous if compared with the length of a convolution filterc, by which the same narrow band could be filtered with the same quality. Indeed, in this case a filter of at least 150–200 points should be used.

The amplitude characteristics of the recursive filter, as suggested by MOONEY, might be improved in the following way. If the pole distances are increased when passing towards the centre of the band to be filtered, a good degree of exclusion could be ensured. On the other hand, the slope could be improved if both extreme poles were placed not radially near the corresponding zero-place, but symmetrically, at the lower limit of the band somewhat to the left of it, at the upper one, to the right of it. In this case, namely, the characteristics of both extreme notch filters overshoot at the proper place, compensating the less steepness of the centre filter. Against the previous five pairs, three zero/pole pairs were placed at each 2° in the 4° band. Thus the number of coefficients was decreased from 19 to 11; at the same time, as shown by the curve *b* of Fig. 2, also the properties of the amplitude characteristics have been improved. The parameters chosen were:

R_p	1,008	1,05	1,008
zero places (φ_0)	34°	36°	38°
poles (φ_p)	33,5°	36°	38,5°

The transfer function:

$$F(z)=\frac{(z-e^{-j34°})(z-e^{+j34°})(z-e^{-j36°})(z-e^{+j36°})(z-e^{-j38°})(z-e^{+j38°})}{(z-1{,}008e^{-j33{,}5°})(z-1{,}008e^{+j33{,}5°})(z-1{,}05e^{-j36°})\times(z-1{,}05e^{j36°})(z-1{,}008e^{-j38{,}5°})(z-1{,}008e^{+j38{,}5°})} \tag{8}$$

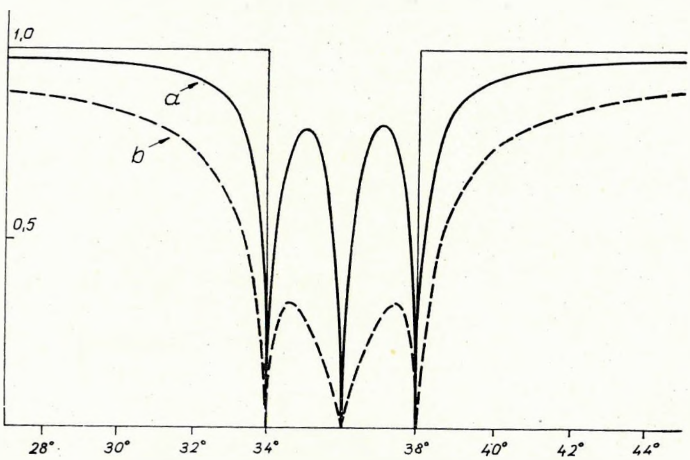

Fig. 4 Amplitude characteristics of band-reject filters. Width of filtered-out band: 4°; zero/pole pairs in intervals of 2°; *a*) $R_p=1{,}008$; *b*) $R_p=1{,}008$; 1,05; 1,008

4. ábra. Sávvágó szűrők amplitúdókarakterisztikái. A kiszűrt sáv szélessége 4°. A zérus—pólus párokat 2°-ként helyeztük el. *a*) $R_p=1{,}008$; *b*) $R_p=1{,}008$; 1,05; 1,008

Рис. 4. Амплитудные характеристики полосно-заграждающих фильтров. Ширина полосы заграждения — 4°. Пары нулевых полюсов размещены через 2°; а) $R_p=1{,}008$ б) $R_p=1{,}008$; 1,05; 1,008

The amplitude characteristics were computed for an $R_p = 1{,}008$, identical with a 2_0 zero/pole pair spacing, in the way suggested by Mooney. The result is shown by curve *a* of Fig. 4. Comparing it with curve *b* of Fig. 2, the former is essentially worse. If only the central R_p distance was increased to 1,05, the characteristics represented by the curve *b* of Fig. 4 were obtained. The exclusion became considerably better, but the steepness decreased at the same time. The conclusion may be drawn that the amplitude characteristics can be improved by a suitable choice of R_p and φ_p values.

The amplitude characteristics presented and the data of the lengths of the necessary weight function show that, in case of narrow-band filtering, recursion filters are more advantageous than convolution filters, the latter being too long in such cases. With an increase of band width, however, also the length of the recursion filter increases, since more and more zero/pole pairs are to be established.

Simultaneously the length of the convolution filter decreases more and more, becoming, in case of broad bands, more advantageous than the recursion filter. In this sense, both procedures are complementing each other. This fact is emphasized and elaborated in details in a forthcoming text-book of Mesko (1971).

The phase distortion of recursion filters

As known, the convolution filters in use are free of phase distortion, but recursion filters are not. Special steps must be taken to make the filtered output free of phase distortion. A simple procedure, known from literature, is, for example, to filter the data system again, from opposite direction. Thus, as additional advantage, the amplitude characteristics will be sharper, while the phase characteristics identically zero. By this procedure, an increase in the number of operations is inevitable, but, recognizing certain symmetry properties of recursion band filters and performing the filtering in two steps, this increase can be kept lower.

Be $X(z)$ the z-transform of the input and $Y(z)$ that of the output. The filter operation in the z-domain is:

$$Y(z) = \frac{A_M(z)}{B_M(z)} X(z). \qquad (9)$$

Performing the filtering in two steps:

$$Y(z) = A_M(z)\,\overline{Y}(z) \qquad (10)$$

where

$$\overline{Y}(z) = \frac{1}{B_M(z)} X(z). \qquad (11)$$

By (10) a convolution filtering, by (11) a recursion filtering is defined, i.e. (9) has been resolved into a convolution and a recursion filtering.

This was done, because it is evident that $A_M(z)$ represents a $2M+1$-point zero-phase-shift convolution filter.

Let us write $A_M(z)$ in the form:

$$A_M(z) = \prod_{i=1}^{M} a_0(z_{0i} - z)(z_{0i} - z). \tag{12}$$

It is known that $A_M(z)$ is of an even degree, since the number of roots is 2 or 4 or 6, etc. Let us write out the i-th pair of roots separately:

$$(z_{0i} - z)(\bar{z}_{0i} - z) = z_{0i}\bar{z}_{0i} - (z_{0i} + \bar{z}_{0i})z + z^2 = 1 - 2z_{0i,\,\text{real}} + z^2. \tag{13}$$

By (13), a 3-point symmetrical operator is represented:

$$a_{i0} = 1 \qquad a_{i1} = -2z_{0i,\,\text{real}} \qquad a_{i2} = 1. \tag{14}$$

In the z-domain, the $A_M(z)$ filter is produced as a product of such factors; in the time-domain, again, as their convolution:

$$A_M = \{1;\ -2z_{01,\,\text{real}};\ 1\} * \{1;\ -2z_{02,\,\text{real}};\ 1\} * \ldots * \{1;\ -2z_{0\frac{M}{2},\,\text{real}};\ 1\}. \tag{15}$$

The convolution of symmetrical filter operators is similarly symmetrical, i.e. in (15)

$$a_i = a_{2M-i} \qquad 0 \leq i \leq M-1 \tag{16}$$

Since $A_M(z)$ is symmetrical, it is free of phase distortion.

This can be very illustratively shown in the z-domain. Let us regard $A_M(z)$ as sum of vectors interpreted on the z-plane. Applying (16), $A_M(z)$ can be written as follows:

$$\begin{aligned} A_M(z) &= z^n(a_0 z^{-n} + \ldots + a_{n-1}z^{-1} + a_n + a_{n-1}z + \ldots + a_0 z^n) = \\ &= z^n(a_0 \bar{z}^n + \ldots + a_{n-1}\bar{z}^1 + a_n + a_{n-1}z + \ldots + a_0 z^n) = \\ &= |A'_M(z)|e^{-j\varphi}\cdot z^n \qquad \sqrt{n} = M \end{aligned} \tag{17}$$

where

$$A'_M(z) = a_0\bar{z}^n + \ldots + a_{n-1}\bar{z}^1 + a_n + a_{n-1}z + \ldots + a_0 z^n \tag{18}$$

the phase-shift:

$$\varphi = \operatorname{arc\,tg} \frac{I_m A'_M(z)}{R_e A'_M(z)}. \tag{19}$$

The z^n factor is considered by shifting the input data by a sampling interval n. It is easy to see that the value is zero. To prove this, one has only to show that the $A_M(z)$ vector has only real component, show well by Fig. 5. The sum of each pair of vectors fall on the real axis, therefore their resultant similarly falls on the same. Thus, Im $A'_1(z) = 0$, and from (19), $\varphi = 0$ for every z.

The situation is not similar in the case of $B_M(z)$, since the poles were not placed on the unit circle, thus the 3-point elementary operators will not be symmetric either.

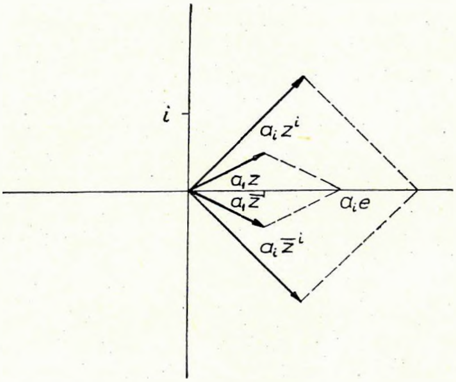

Fig. 5 The $A'_M(z)$ vector in the z-domain
5. ábra. Az $A'_M(z)$ vektor z tartományban
Рис. 5. Вектор $A'_M(z)$ в области z

It is visible, therefore, that phase-distortion is caused only by a recursion filtering with a filter $\frac{1}{B_M(z)}$. Merely this has to be corrected, for example in the way mentioned earlier.

The original data system is passed twice across the filter $\frac{1}{B_M(z)}$ from opposite directions, then filtered in a non-recursive way with $A_M(z)$. The difference is that the data system had been filtered, until now, with the entire $\frac{A_M(z)}{B_M(z)}$ filter, unnecessarily. The block diagram of the filtering suggested is visible in Fig. 6.

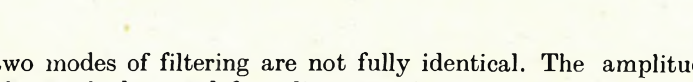

Fig. 6 Block diagram of the filtering suggested
6. ábra. A javasolt szűrés blokksémája
Рис. 6. Предлагаемая схема фильтрации

These two modes of filtering are not fully identical. The amplitude characteristics of the method quoted from literature is

$$|F_1(W)| = \left|\frac{A_M(z)}{B(z)}\right|^2_{|z=e^{-jW}} = \frac{|A_M(z)|}{|B_M(z)|^2} \cdot |A_M(z)|_{|z=e^{-jW}} \qquad (20)$$

while that of the one suggested:

$$|F_2(W)| = \frac{|A_M(z)|}{|B_M(z)|^2}_{|z=e^{-jW}} . \qquad (21)$$

The difference lies in the factor $A_M(z)$. In case of suitably chosen parameters, however, both approach the ideal characteristics fairly well.

Also the number of operations spared was estimated. If the number of established pairs of roots is n, i.e. $A_M(z)$, resp. $B_M(z)$ polynomials are of $M = 2n$-th degree, the number of operations needed for each output point is less by $4n + 1$. Let us assume that a channel of length of $2K$ is filtered, and $n = 6$. The gain is about 51 000 operations. It must be noted that this estimate does not take the increase in computer time, originating from the more complicated data motion, into consideration.

The situation will be somewhat altered when the lower limit of the band to be filtered is zero, i.e. when a low-pass filter is designed. In this case $A_M(z)$ will not be symmetrical, its radical form will be broadened by factor $(c - z)$, filtering out the zero frequency. Here c is real. This corresponds to a non-symmetrical operator $(c_i - 1)$, and the symmetry of the numerator will be destroyed by the convolution made with it. Accordingly, this factor must be detached from $A_M(z)$. The filtering will be modified in such a way that a two-way recursive filtering is made with $\frac{c-z}{B_M(z)}$ according to the formula

$$F(z) = \frac{A_M(z)}{B_M(z)} = A_1(z)\,\frac{c-z}{B_M(z)} \tag{22}$$

where

$$A_M(z) = A_1(z)\cdot(c-z)$$

then the convolution filter $A_1(z)$ is applied to the result.

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GÖNCZ GÁBOR—ZELEI ANDRÁS

REKURZÍV SÁVSZŰRŐK TERVEZÉSE

A tanulmány első részében rövid áttekintést ad a rekurzív sávszűrőkről, alkalmazásuk előnyeiről és korlátairól, valamint tervezésük egyfajta — az irodalomból jól ismert — módjáról. A módszer alkalmazásával a szűrők amplitúdókarakterisztikája javítható. A rekurzív szűrők fáziskarkterisztikájára vonatkozólag ismeretes, hogy ezen szűrők nem zérus fázistolásúak. Bizonyos eljárással — az egyszer már megszűrt adatok ismételt szűrésével — a fázistorzítást ki lehet küszöbölni. Figyelembe véve a rekurzív szűrők átviteli függvényének bizonyos szimmetria tulajdonságait ez a második szűrés lerövidíthető.

Г. ГЁНЦ—А. ЗЕЛЕИ

РАЗРАБОТКА РЕКУРСИВНЫХ ПОЛОСНЫХ ФИЛЬТРОВ

В первой части работы коротко излагаются существующие виды рекурсивных полосных фильтров, их преимущества и ограничения, а также хорошо известный из литературы метод их разработки. Метод позволяет улучшить амплитудные характеристики фильтров. О фазовой характеристике рекурсивных фильтров известно, что в них фазовое смещение не равняется нулю. С применением определенного способа — путем повторной фильтрации уже отфильтрованных данных — можно исключить фазовые искажения. Учитывая определенные особенности симметрии характеристики рекурсивных фильтров можно сократить вторую фильтрацию.

Függelék

A függelékben a rekurzív szűrők zérus—pólus elhelyezéssel történő tervezését kívánjuk szemléletessé tenni néhány egyszerű példa segítségével.

Tegyük fel, hogy olyan lyukszűrőt akarunk tervezni, z-tartományban, mely a zéró frekvenciát, azaz az egyenkomponenst szűri ki a bemeneti adatrendszerből. Jelöljük ezen szűrő átviteli függvényét $F(z)$-vel. Minthogy definíció szerint $z = e^{-j2\pi f\tau}$ — ahol τ a mintavételi távolság — olyan $F(z)$-t kell választanunk, melyre fennáll, hogy $F(1) = 0$. Ez teljesül ha:

$$F(z) = \frac{1-z}{2}. \tag{1}$$

Az átviteli függvény zéróhelye $z = 1$-nél van. Az amplitúdókarakterisztika:

$$|F(z)| = \frac{|1-z|}{2}. \tag{2}$$

Az 1. ábrán látható, hogy az $(1-z)$ vektor hossza egyenlő az amplitúdókarakterisztika valamely z helyen felvett értékével. Látható, hogy $F(1) = 0$ és $F(-1) = 1$.

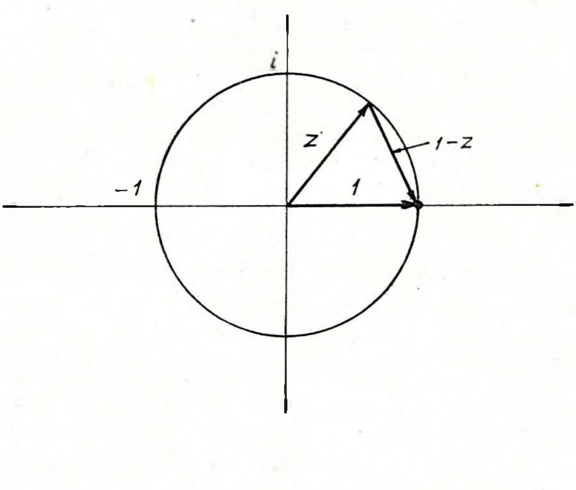

Fig. 1 Transfer function of notch filter $F(z)$ on the z-plane

1. ábra. Az $F(z)$ lyukszűrő átviteli függvénye a z-síkon

Рис. 1. Характеристика фильтра-пробки $F(z)$ на плоскости z

A 2. ábra *a* görbéje mutatja az amplitúdókarakterisztikát a Nyquist-intervallumban. Ez, távolodva a zérus frekvenciától, igen lassan növekszik, emiatt erősen vágja a zérus frekvencia elég tág környezetét is. Ez nemkívánatos hatás. Mindenesetre az (1; −1) kétpontos szűrő megvalósítja az egyenkomponens kiszűrését. Élesebb amplitúdókarakterisztikájú szűrőre lenne azonban szükségünk. Definiáljuk $G(z)$-t a következőképpen:

$$G(z) = 2\frac{F(z)}{R_p - z}, \tag{3}$$

ahol R_p valós.

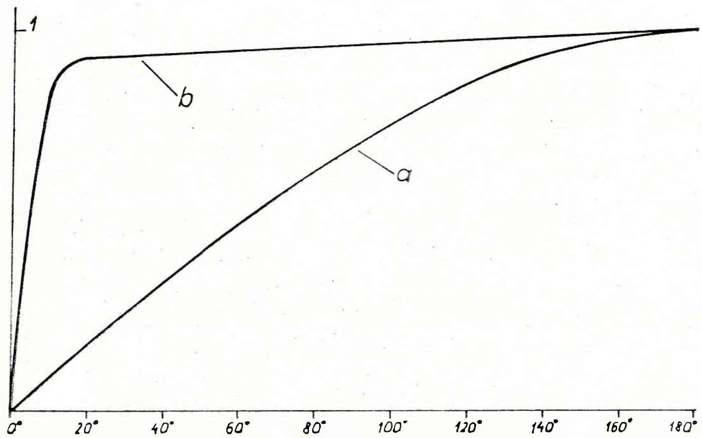

Fig. 2 Amplitude characteristics of filter $F(z)$ (curve *a*) and of $G(z)$ (curve *b*)

2. ábra. Az *a*-val jelölt görbe az $F(z)$ a *b*-vel jelölt a $G(z)$ szűrő amplitúdókatakterisztikája

Рис. 2. Амплитудные характеристики а) фильтра $F(z)$; б) фильтра $G(z)$

$G(z)$ zéróhelye megegyezik $F(z)$ zéróhelyével, és pólusa van a $z = R_p$ helyen. A 3. ábra szemlélteti az elhelyezett pólus hatását. A $|G(z)|$ az $(1 - z)$ és a $(R_p - z)$ vektorok hosszának hányadosa. Ha $R_p = 1 + a$, ahol $0 < a \ll 1$ akkor $|1 - z|$ és $|R_p - z|$ közelítőleg egyenlő és így $\frac{|1 - z|}{|R_p - z|} \simeq 1$ minden z-re, kivéve a $z = 1$ — azaz az $f = 0$ — helyet, ahol $|G(1)| = 0$. Az ábráról leolvasható a fáziskarakterisztika értéke is:

$$\varphi(z) = \Phi(z) - \psi(z). \tag{4}$$

A 2. ábra *b* görbéje a $G(z)$ szűrő amplitúdókarakterisztikája. Látható tehát, hogy egy zérushely és egy pólus elhelyezésével igen nagy vágási meredekségü szűrőt állíthatunk elő.

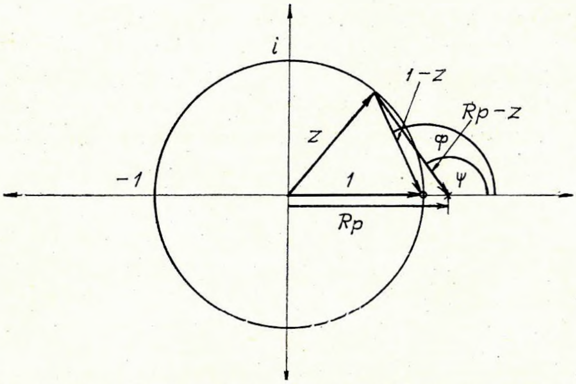

Fig. 3 Transfer function of filter $G(z)$ on the z-plane

3. ábra. A $G(z)$ szűrő átviteli függvénye a z-síkon

Рис. 3. Характеристика фильтра $G(z)$ на плоскости z

Ha $X(z)$-vel jelöljük a bemenet, $Y(z)$-vel a kimenet z-transzformáltját, a szűrést z-tartományban a következő képlet írja le:

$$Y(z)=\frac{\frac{1}{R_p}-\frac{1}{R_p}\cdot z}{1-\frac{1}{R_p}\cdot z}\cdot X(z), \tag{5}$$

időtartományban pedig:

$$Y_n=\frac{1}{R_p}X_n-\frac{1}{R_p}X_{n-1}+\frac{1}{R_p}Y_{n-1} \tag{6}$$

a szűrő hárompontos, R_p értékeire a szokásos választás $1{,}01\leq R_p\leq 1{,}1$ ettől függően változik a lyukszűrő meredeksége. Bonyolultabb szűrőket is hasonló meggondolásokkal tervezhetünk.

Felvetődhet a kérdés miért nem tehetők pólusok az egységkör belsejébe. Általánosan bizonyítható, hogy az ily módon kapott szűrő nem lenne stabil. Ismeretes, hogy egy lineáris szűrő akkor stabil, ha az egységimpulzus bemenetre adott válasz — amelyet súlyfüggvénynek is szokás nevezni — időben lecseng, azaz ha a rendszer visszatér nyugalmi állapotába. Más szavakkal, ha véges energiájú bemenet esetén a kimenet energiája is véges lesz.

Vizsgáljuk meg a fenti szűrő stabilitását. Ehhez szükségünk lesz az egységimpulzus bemenet z-transzformáltjára. Az egységimpulzus bemenet:

$$X_n=\begin{cases}1 & n=0\\ 0 & n\neq 0\end{cases}. \tag{7}$$

A z-transzformált definíciója:

$$X(z)=\sum_{n=0}^{\infty}X_n z^n. \tag{8}$$

Így (7) z-transzformáltja $X(z)=1$.

A kimenet z-transzformáltja (5) alapján:

$$Y(z)=\frac{\frac{1}{R_p}-\frac{1}{R_p}\cdot z}{1-\frac{1}{R_p}\cdot z}\,X(z)=\frac{\frac{1}{R_p}-\frac{1}{R_p}\cdot z}{1-\frac{1}{R_p}z}, \tag{9}$$

amely éppen az átviteli függvény, a $|z|=1$ körön.

(9)-et sorbafejtve $Y(z)$-re a következő képletet kapjuk:

$$Y(z)=1+\left(\frac{1}{R_p}-1\right)z+\left(\frac{1}{R_p^2}-\frac{1}{R_p}\right)z^2+\ldots+\left(\frac{1}{R_p^n}-\frac{1}{R_p}\right)z^n+\ldots \tag{10}$$

a kimenet időtartományban tehát az

$$1;\ \frac{1}{R_p}-1;\ \frac{1}{R_p^2}-\frac{1}{R_p};\ \ldots;\ \frac{1}{R_p^n}-\frac{1}{R_p^{n-1}};\ \ldots \tag{11}$$

sorozat lesz.

Ha $R_p=1$ a kimenet megegyezik a bemenettel, ekkor ugyanis $G(z)=1$ lesz.

Ha $R_p>1$ akkor:

$$\lim_{n\to\infty}\left(\frac{1}{R_p^n}-\frac{1}{R_p^{n-1}}\right)=(1-R)\lim_{n\to\infty}\left(\frac{1}{R_p}\right)^n=0$$

tehát a kimenet lecseng.

Végül ha $R_p<1$ akkor $\frac{1}{R_p}>1$, így a kimenet minden határon túl nő. A szűrő instabil.

A pólusokat tehát mindenkor az egységkörön kívül kell elhelyezni.

A zérus és pólushelyek az előző példákban a valós tengelyen voltak. Általában ezek komplex számok. Ilyen esetekben a megfelelő komplex konjugált értékeket is figyelembe kell venni a szűrő tervezésénél.