Convexity and Mathability

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Abstract: Mathability refers to a branch of cognitive infocommunications that investigates any combination of artificial and natural cognitive capabilities relevant to mathematics, including a wide spectrum of areas ranging from low-level arithmetic operations to high-level symbolic reasoning. In connection with investigations related to mathability and to applications of computer-assisted methods for studying mathematical problems, in this paper, animation of the planar hyperconvex sets of radius r is presented. This animation helps us understand some properties of hyperconvex sets and to see the differences between convexity and hyperconvexity.

Keywords: Mathability; Cognitive Infocommunications; Computer Assisted Methods; Animation; Convex set; Hyperconves set of raduis r; Spindle convex set; Ball polyhedron

1 Introduction

Mathability refers to a branch of cognitive infocommunications that investigates any combination of artificial and natural human cognitive capabilities relevant to mathematics, including a wide spectrum of areas ranging from low-level arithmetic operations to high-level symbolic reasoning. The concept was introduced in the paper [1] related to the 4th IEEE International Conference on Cognitive Infocommunications (CogInfoCom) in 2013. Mathability refers to devices with high mathematical and logical potential and is defined as human mathematical ability [8]. Mathability mainly discusses what new assimilation methods are used to process information and how people use this ability to build their knowledge using problem-solving and experiences as well as high-level mathematics applications [8]. Its educational aspects were investigated, among others, in [5]-[13], while [14]-[18] papers focus on human cognitive related aspects of CogInfoCom and how people can communicate with machines to possess new knowledge. Questions related to mathability and to computer-based methods for investigations of mathematical problems have been studied by several authors during recent years [19]-[23]. Computer-aided solutions of mathematical problems were presented in [22], [26] and some of its further general properties were described in the papers [2] and [3] and in the book [4]. In this paper, we also would like to contribute to these investigations. We present a computer-assisted method for a visualization related to the so-called hyperconvex discs of radius r.

Although convexity is one of the oldest concepts in geometry, it is used to investigate some modern phenomena in mathematics, i.e. this property is used in the qualitative theory of differential equations as well [24], [25].

The students meet the convexity several times during the education. In this paper, the basics of convexity and some generalizations of convexity will be introduced. The paper [26] motivated us to write an animation of hyperconvex discs of radius r. The animation is developed in GeoGebra available at https://www.geogebra.org/.

The hyperconvexity is a generalisation of convexity. Such kinds of generalisation of convexity shoves us the deeper attributes of convexity. The presented animation helps us understand some properties of hyperconvex sets and to see the differences between convexity and hyperconvexity.

2 The Convexity

The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The notation $\cap$ means the intersection of sets. The points and vectors are identified in a natural way. In this paper, xy will also denote the length of the segment xy.

In this section we write the basic concepts of convexity.

In the school of geometry, a figure is called convex if it contains all segments if the endpoints of the segments lie in the figure. The next definition is the same.

Definition 2.1. The set C is *convex* if $x, y \in C$ implies that for any $\lambda \in [0,1]$ we have

$$\lambda x + (1-\lambda)y \in C. \qquad (1)$$

Definition 2.2. The set N is *non-convex* if it is not convex, i.e. there are at least two points $x, y \in N$, and a λ number ($\lambda \in [0,1]$) such that

$$\lambda x + (1-\lambda)y \notin N. \qquad (2)$$

Example 2.1. In Figure 1 we can see a convex set (left) and a non-convex set (right). Indeed, the right set is not convex, the midpoint m of the segment pq does not lie in the set.

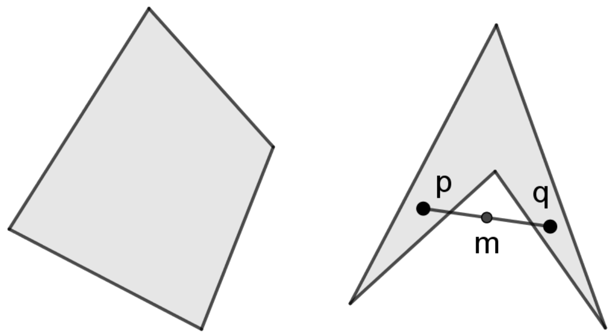

Figure 1
A convex set (left) and a non-convex set (right)

Example 2.2. The empty set is convex. The straight line segment is convex. The whole plane is convex.

The convex combination of the two points x and y is the set $\lambda x + (1 - \lambda)y$ for all $\lambda \in [0,1]$ which is the straight line segment connecting the points x and y. The convex combination of finitely many points is the following.

Definition 2.3. The *convex combination* of the points $x_1, \dots, x_k \in \mathbb{R}^n$ is the linear combination

$$\lambda_1 x_1 + \cdots + \lambda_k x_k \tag{3}$$

for

$$\lambda_1 \geq 0, \dots, \lambda_k \geq 0 \text{ and } \lambda_1 + \cdots + \lambda_k = 1. \tag{4}$$

Example 2.3. The set of all convex combinations of two different points is a straight line segment. The set of all convex combinations of three non-collinear points is a triangle.

Remark 2.1. The set C is convex if and only if all the convex combinations of the points $x_1, \dots, x_k \in \mathbb{R}^n$ lie in the set C.

Theorem 2.1. The intersection of convex sets is convex.

This implies the next definition.

Definition 2.4. The *convex hull* of the set $S \in \mathbb{R}^n$ is the intersection of all convex sets containing S.

Remark 2.2. The convex hull of a set is convex.

Example 2.4. The convex hull of a convex set C is C.

Example 2.5. The convex hull of three different non-collinear points x_1, x_2, x_3 is the triangle of vertices x_1, x_2, x_3.

Definition 2.5. The *Minkowski sum* of the sets $S_1, S_2 \in \mathbb{R}^n$ is the set

$$\{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}. \quad (5)$$

Notation: $S_1 + S_2$.

Example 2.6. The Minkowski sum of two non-parallel segments is a parallelogram.

Theorem 2.2. Let C_1 and C_2 be two convex sets in R^n. The translation $C_1 + p$ is convex. The scaling αC_1 is convex. The orthogonal projection of the set C_1 is convex. The Minkowski sum $C_1 + C_2$ is convex.

3 Some Generalizations of Convexity

3.1 The m-convexity

Toader [27] introduced the m-convexity in the following way.

Definition 3.1.1. Let $m \in [0,1]$ be a fixed number. The set $C \in R^n$ is m-convex if

$$tx + m(1-t)y \in C \quad (6)$$

for all elements $x, y \in C$ and for each $t \in [0,1]$.

Example 3.1.1. If $m = 1$, then the definitions convex and m-convex are the same.

It is a consequence of the definition, that if $m \neq 1$, then it is necessary to consider the origin as well.

Example 3.1.2. If $m = 0.5$, then the m-convex set containing the points x and y contains the point $\frac{m}{m+1}(x + y)$ as well (see, e.g. [26])

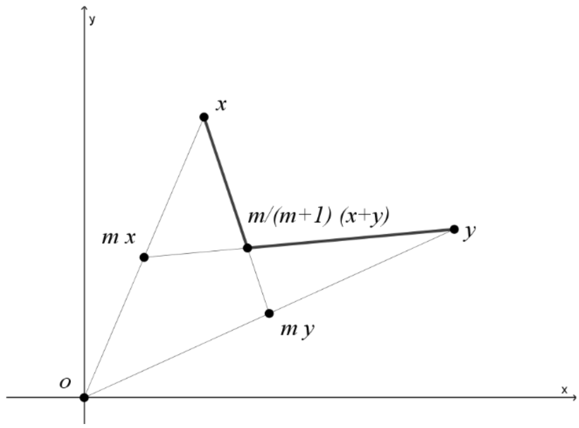

Figure 2
The line connecting the points x and y in an m-convex set

Definition 3.1.2. The *m-convex combination* of the points $x_1, \dots, x_k \in \mathbb{R}^n$ is the linear combination

$$\lambda_1 x_1 + m(\lambda_2 x_2 + \cdots + \lambda_k x_k) \tag{7}$$

for

$$\lambda_1 \geq 0, \dots, \lambda_k \geq 0 \text{ and } 0 < \lambda_1 + \cdots + \lambda_k \leq 1. \tag{8}$$

Definition 3.1.3. The *m-convex hull* of the set $S \in \mathbb{R}^n$ is the intersection of all m-convex sets containing S.

Theorem 3.1.1 [28] The set S is m-convex if and only if S is the set of all m-convex combinations of points lying in S.

Theorem 3.1.2 [28] The m-convex hull of the set S is the set of all m-convex combinations of points lying in S.

Theorem 3.1.3 [28] Let S be a set containing the origin o. The set S is m-convex if and only if for all $x, y \in S$ the set $conv\left(o, x, \frac{m}{m+1}(x+y)\right) - \{o\}$ is contained in S.

Example 3.1.3. The m-convex hull of the two different points x, y and the origin o is the (degenerate) quadrangle $ox\frac{m}{m+1}(x+y)y$ (Figure 2).

In [21] and [25] we can find an animation of m-convex hull of finitely many points if m is varied. This paper motivated us to produce a similar animation for hyperconvex sets of radius r.

3.2 The Hyperconvex Sets of Radius r

Definition 3.2.1. The *n-dimensional ball* (or shortly *n-ball*) of radius r and center c in R^n, denoted by $B(r,c)$, is $\{x \in R^n : xc \leq r\}$. If $n = 2$, then the ball is called *disc*.

Definition 3.2.2. The *n-dimensional sphere* (or shortly *n-sphere*) of radius r and center c in R^n is $\{x \in R^n : xc = r\}$. If $n = 2$, then the sphere is called *circle*.

Definition 3.2.3. Let $x, y \in R^n$. If $xy < 2r$, then the *spindle of radius* r (or shortly *spindle) of x and y* is defined as the union of circular arcs with endpoints x and y that are of radii at least r and shorter than a semicircle of radius r. If $xy = 2r$, then the *spindle* of *x and y* is defined as the disc of radius r and center $(x + y)/2$. If $xy > 2r$, then the *spindle* of *x and y* is defined as $\emptyset$.

Remark 3.2.1. The spindle of x and y is the intersection of the balls of radii r and containing x and y.

Example 3.2.1. In Figure 3 can be found a spindle of radius 1 of x and y on the plane if $xy < 2$.

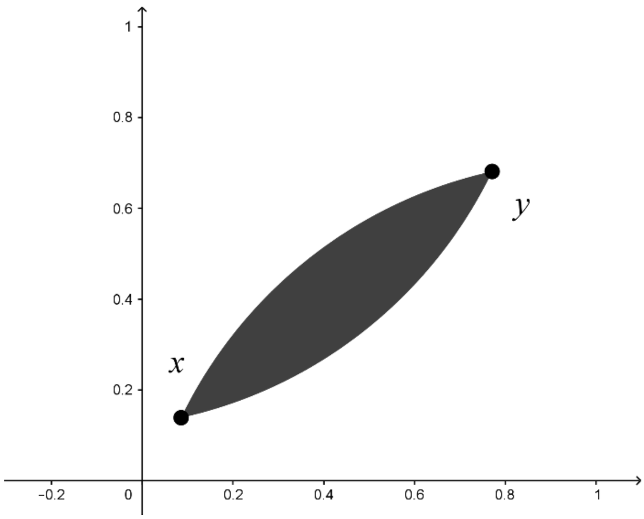

Figure 3
The spindle of x and y

Definition 3.2.4. Let C be the set such that the diameter of C is less than or equal to $2r$. The set C is *spindle convex* (of radius r) if $x, y \in C$ implies that the spindle of x and y is a subset of C.

Definition 3.2.5. The *circumradius*, denoted by $cr(C)$ of a bounded set C in R^n is defined as the radius of the unique smallest ball that contains C. If C is unbounded, then $cr(C) = \infty$.

Definition 3.2.6. A set C in R^n is *hyperconvex of radius* r (or shortly *hyperconvex*) if is the intersection of n-balls of radius r.

Remark 3.2.2. Observe if we consider half-spaces as balls of infinite radius, then the hyperconvexity of radius r and (linear) convexity are the same.

Definition 3.2.7. Let C be a finite set in R^n such that $cr(C) \leq r$. The *ball-polyhedron of radius* r (or shortly *ball-polyhedron*) (generated by C) is the intersection of the balls for radii r and centers of points in C. If $n = 2$, then a ball-polyhedron is called a *disk-polygon*.

Observe the ball-polyhedron of radius r generated by C is

$$P = \bigcap_{c \in C} B(r, c). \tag{9}$$

First Mayer [29] considered ball-polyhedra in 1935 and called this property "überkonvex". Mayer's paper inspired several researchers in the first half of the 20th Century e.g. [30]-[36]. 1980's we can find this property as r-convex or spindle convex of hyperconvex of radius r see, e.g. [37]-[47].

Definition 3.2.8. If a ball B contains a set C in R^n and a point x lies on the boundary of B and the boundary of C at the same time, then B supports C at x.

Theorem 3.2.1. Let C be a closed convex set in R^n such that $cr(C) \leq r$. The following are equivalent.

1) The set C is spindle convex of radius r.
2) The set C is the intersection of unit balls of radius r containing C.
3) For every boundary point of C, there is a ball of radius r that supports C at that point.

Definition 3.2.9. The *hyperconvex hull of radius* r of the set $S \in \mathbb{R}^n$ is the intersection of all hyperconvex sets of radius r containing S.

4 The Description of Animation

We use the dynamic free software GeoGebra, which can be downloaded from https://www.geogebra.org/.

To have a GeoGebra file, which can be easily modified, we use scripts under buttons.

The first GeoGebra script under button1 in On Click is the input of points, a list which consists of the points, the default value of r, and a text. In this special case, we use six points.

```
P1=(20,5); P2=(13,6.5); P3=(17,13); P4=(12,16); P5=(7,17); P6=(0.5,7)
L={P1,P2,P3,P4,P5,P6}
r=1
text1= "The diameter is larger than 2r."
```

The JavaScript under button2 in On Click is the drawing of the hyperconvex set of radius r generated by the points in the list L. The code is the following.

```
1 D=ggbApplet.getValue("length(L)");
2 ggbApplet.evalCommand("LSegm_{0}={}"); k=1;
3 for(var i =1;i<D+1;i++) for(var j=i+1;j<D+1;j++) {
4 ggbApplet.evalCommand("LSegm_{"+k+"}=Append(LSegm_{"+(k-
1)+"},Segment(L("+i+"),L("+j+")))");
5 k=k+1; }
6 ggbApplet.evalCommand("Diam=Max(LSegm_{"+(k-1)+"})");
7 ggbApplet.evalCommand("Conv=ConvexHull(L)");
8 ggbApplet.evalCommand("ShowLabel(Conv,False)");
9 for(var i =1;i<D+1;i++) {
10 ggbApplet.evalCommand("C_{"+i+"}=Circle[L("+i+"),r]");
11 ggbApplet.evalCommand("SetVisibleInView(C_{"+i+"},1,False)"); }
12 for(var i =1;i<D+1;i++) for(var j =i+1;j<D+1;j++) {
13 ggbApplet.evalCommand("Center_{"+i+","+j+",1}=Intersect[C_{"+i+"}, C_{"+j+"},1]");
14 ggbApplet.evalCommand("SetVisibleInView(Center_{"+i+","+j+",1},1, False)");
15 ggbApplet.evalCommand("Center_{"+i+","+j+",2}=Intersect[C_{"+i+"}, C_{"+j+"},2]");
16 ggbApplet.evalCommand("SetVisibleInView(Center_{"+i+","+j+",2},1, False)");
17 ggbApplet.evalCommand("ineq_{"+i+","+j+",1}=(x-x(Center_{"+i+","+j+",1}))^2+(y-
y(Center_{"+i+","+j+",1}))^2-r^2<=0");
18 ggbApplet.evalCommand("SetVisibleInView(ineq_{"+i+","+j+",1},1, False)");
19 ggbApplet.evalCommand("ineq_{"+i+","+j+",2}=(x-x(Center_{"+i+","+j+",2}))^2+(y-
y(Center_{"+i+","+j+",2}))^2-r^2<=0");
20 ggbApplet.evalCommand("SetVisibleInView(ineq_{"+i+","+j+",2},1, False)");
21 ggbApplet.evalCommand("ineq_{"+i+","+j+"}=ineq_{"+i+","+j+",1}&&
ineq_{"+i+","+j+",2}");
22 ggbApplet.evalCommand("SetLineThickness(ineq_{"+i+","+j+"},0)");
23 ggbApplet.evalCommand('SetColor(ineq_{'+i+','+j+'},"#000000")');
24 ggbApplet.evalCommand("ShowLabel(ineq_{"+i+","+j+"},False)");
25 ggbApplet.evalCommand("Delete(eq1)");
26 ggbApplet.evalCommand("Delete(eq2)"); };
27 ggbApplet.evalCommand("If(Diam>2*r,SetVisibleInView(text1,1,
True),SetVisibleInView(text1,1,False))");
```

This is a rough algorithm. Our aim was visualization and not effectiveness. The first line adds the value of the length of the list L to the JavaScript code. Lines 2-6 determine the diameter of the point set lying in the list L. Since GeoGebra is a dynamic language the modification of the points lying in L implies the modification of the diameter calculated by this code in real-time. Lines 7-8 produce the convex hull of the point set lying in the list L. Lines 9-26 define the spindles of the pairs of points in the list L. Line 27 hides the text text1 if the diameter of L is less than or equal to 2r. The result is the hyperconvex hull of radius r generated by the points in L. If we let show the object r as a slide between 0.01 and 5, then we can vary the radius r in the hyperconvex set. If we switch on the animation of r, then in Figures 4 and 5 we can see a selection of six stages of the hyperconvex hull of radius r of the list of points given for the values $r = 1, 1.2, 1.4, 1.6, 1.8$ and 5.

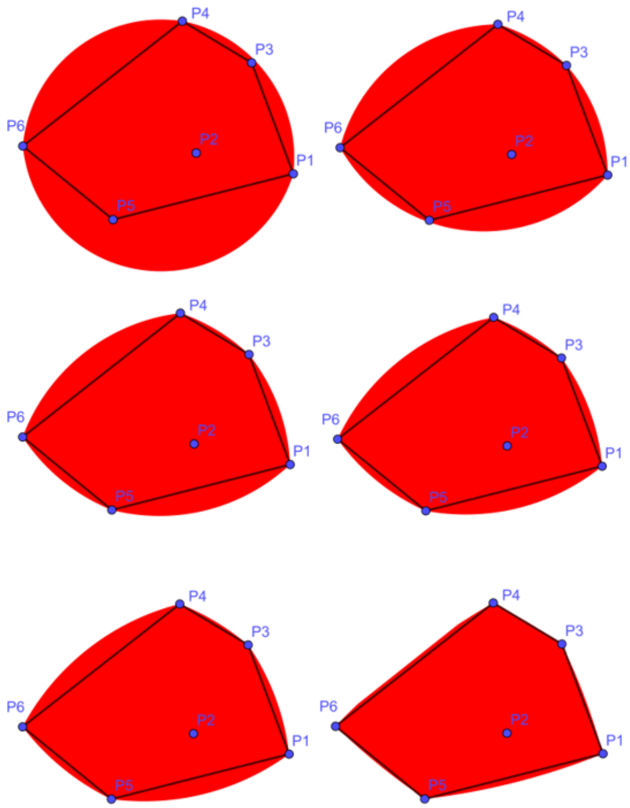

Figure 4

Some stages from the animation

Conclusion

The results presented here are connected to the investigations of mathability (cf. [1] and [4].) Nowadays hyperconvexity is a popular generalisation of convexity in the literature of discrete and convex geometry (see e.g. [43]).

Convexity is a wide range applicable concept in mathematics. The first step considering hyperconvex sets is drawing such a set on a piece of paper. This dynamic animation enables us to draw this figure. In order to imagine a hyperconvex set in higher dimensions we have to understand the planar case.

Since a generalization of convexity can be challenging to imagine or understand, it is important to visualize hyperconvexity. We can find serious theorems in the literature considering the difference between linear convexity and a generalization of convexity. If we use such simple animations, then we can make conjecture about new theorems and about the difference between the two kinds of convexity easier.

The presented method inspires us to visualize geometric properties of point sets in GeoGebra. GeoGebra supports the script commands as we can see in this paper. It could be a different opportunity to create a new tool in GepGebra to visualize the hyperconvex hull of a finite point set.

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