

Discrete Duality Finite Volume Scheme for the Curvature-driven Level Set Equation

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Abstract: There are many approaches to obtain the numerical solution of the curvature driven level set equation [7]. They are based on the finite differences method [7], or the finite element method [3] or the finite volume method [5]. The proposed numerical scheme uses the so called discrete duality finite volumes as in [1].

Keywords: curvature driven level set equation; discrete duality finite volume method

1 Introduction

The curvature driven level set equation [7]

$$u_t - |\nabla u| \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = 0 \quad (1)$$

is used in many different applications – the motion of interfaces, crystal growth in thermomechanics and computational fluid dynamics, the smoothing and segmentation of images and the surface reconstructions in image processing.

The derivation of our numerical method for solving equation (1) is based on the finite volume methodology. We construct the so-called discrete duality finite volume scheme. Computational scheme follows from weak (integral) formulation of integrating equation (1) on the volume.

2 Semi-Implicit Discrete Duality Finite Volume Scheme

Let have a digital image given on a structure of pixels of (in general) rectangular shape. And let the unknown function $u(t, x)$ in (1) be given by discrete values in the centers of these rectangles. The set of all these pixels can create the original mesh. Let us create the dual mesh. The new unknown function $v(t, x)$ will be given by discrete values in the corners of original rectangles (see Figure 1).

Unknown function u will be defined on $Q_T = I \times \Omega$, where Ω can be two dimensional bounded Lipschitz domain (in our case very often a rectangular) and $I = [0, T]$ is the time interval. We will consider zero Neumann boundary conditions (2) and an initial condition (3) of the type:

$$\partial_\nu u = 0 \text{ on } I \times \partial\Omega, \quad (2)$$

$$u(0, x) = u^0(x). \quad (3)$$

A discrete time step is needed for the numerical scheme construction. We will consider the uniform time step $\tau = \frac{T}{N}$, where N is a total number of filtering steps, and the time derivative in (1) will be replaced by backward difference $\frac{u^n - u^{n-1}}{\tau}$. Semi-implicitness consists in computing the nonlinear terms values from the previous time step and for linear ones we use values from the current time level.

Semi-implicit in time discretization: Let τ be a given time step, and u^0 be a given initial level set function. Then, for $n = 1, \dots, N$, we look for a function u^n , solution of the equation

$$\frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} = \nabla \cdot \left(\frac{\nabla u^n}{|\nabla u^{n-1}|} \right). \quad (4)$$

Now we will tell something about fully discrete scheme. As we mentioned above, a digital image is given on a structure of pixels with rectangular shape (red rectangles in Figure 1). This set of pixels can represent original rectangular finite volume mesh. We denote it by τ_h .

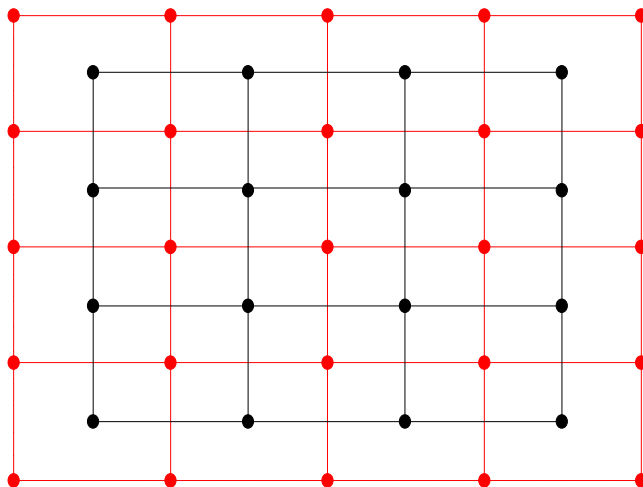


Figure 1

Original (red rectangles) and dual (black rectangles) mesh

Since in every discrete time step of the method for (4) we have to evaluate the gradient of the level set function at the previous step $|\nabla u^{n-1}|$, we put a diamond-shaped regions to edge onto the computational domain and then take an approximation by finite differences on these regions. Thus we obtain a simple and fast construction of a system of equations. The values of the unknown functions u and v are given by values in the centers of the original and dual volumes, respectively. To derive the numerical scheme, we use the same notation as in [2].

Our original volume mesh will consists of cells $V_{ij} \in \tau_h$, associated with DF nodes x_{ij} , say $i = 1, \dots, N_1, j = 1, \dots, N_2$. Dual mesh, shifted to the north-east direction, consist of cells $\bar{V}_{ij} \in \bar{\tau}_h$ associated only with DF nodes \bar{x}_{ij} , say $i = 1, \dots, N_1, j = 1, \dots, N_2$ in such a way that \bar{x}_{ij} is the right top corner for the volume V_{ij} of original mesh.

We describe all notations only for the original volume mesh. For the dual mesh the notation will be the same, but with "overlines" and the unknown function will be denoted by v . We will consider the set of all neighbouring nodes (north, south, west, east) $V_{i+p, j+q}, p, q \in \{-1, 0, 1\}, |p| + |q| = 1$ for each volume $V_{ij} \in \tau_h$ and denote it by N_{ij} . Let $m(V_{ij})$ represent the measure of V_{ij} . The segment connecting the center of V_{ij} and the center of its neighbour $V_{i+p, j+q} \in N_{ij}$ is

denoted by σ_{ij}^{pq} and its length by h_{ij}^{pq} . As we have regular rectangular volume mesh, we can use shorter notations h_1 for h_{ij}^{p0} , $p \in \{-1, 1\}$, h_2 for h_{ij}^{0q} , $q \in \{-1, 1\}$ representing the sizes of the finite volumes in x_1, x_2 directions, respectively. The sides of the finite volume V_{ij} are denoted by e_{ij}^{pq} with measure $m(e_{ij}^{pq})$. The segment σ_{ij}^{pq} connecting the centers of V_{ij} and the center of its neighbour $V_{i+p, j+q} \in N_{ij}$ crosses the side e_{ij}^{pq} of volume V_{ij} in the point x_{ij}^{pq} . We will use the notation $u_{ij} = u_h(x_{ij})$, where x_{ij} is the center of the volume V_{ij} and also $u_{ij}^n = u_{h,\tau}(x_{ij}, t_n)$, which means $u_{h,\tau}(x_{ij}, t_n)$ is piecewise constant function in space and time.

Let us integrate (4) over every finite volume V_{ij} , $i = 1, \dots, N_1$, $j = 1, \dots, N_2$, and then using a divergence theorem we get an integral formulation of (4)

$$\int_{V_{ij}} \frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} dx = \sum_{|p|+|q|=1} \int_{e_{ij}^{pq}} \frac{1}{|\nabla u^{n-1}|} \frac{\partial u^n}{\partial \nu} ds \quad (5)$$

where ν is a unit outer normal to the boundary of V_{ij} . Now we have to approximate numerically the exact "fluxes" on the right hand side and the "capacity function" $\frac{1}{|\nabla u^{n-1}|}$ on the left-hand side. For the approximation of the left-hand side of (5) we get

$$\int_{V_{ij}} \frac{1}{|\nabla u^{n-1}|} \frac{u^n - u^{n-1}}{\tau} dx \approx \frac{m(V_{ij})}{\bar{Q}_{ij}} \frac{u_{ij}^n - u_{ij}^{n-1}}{\tau} \quad (6)$$

where \bar{Q}_{ij} is an average modulus of gradient in V_{ij} .

This average will be computed using the values of the gradients on each side e_{ij}^{pq} of the finite volume, which we must approximate on the right hand of (5) as well. The normal derivative is naturally expressed by the finite difference of the neighbouring pixel values divided by the distance between pixel centers. To approximate modulus of gradients on pixel sides, we use following definitions for $p, q \in \{-1, 0, 1\}$, $|p| + |q| = 1$ and $\alpha(p) = 0$ if $p \geq 0$, and $\alpha(p) = -1$ if $p = -1$.

$$\nabla^{p0} u_{ij}^n = \left(p(u_{i+p,j}^n - u_{i,j}^n) / h_1, (v_{i+\alpha(p),j} - v_{i+\alpha(p),j-1}) / h_2 \right) \quad (7)$$

$$\nabla^{0q} u_{ij}^n = \left((v_{i,j-\delta(q)} - v_{i-1,j-\alpha(q)}) / h_1, q(u_{i,j+q}^n - u_{ij}^n) / h_2 \right) \quad (8)$$

Specially we have:

$$(p, q) = (1, 0) : \nabla^{pq} u_{ij}^n = \left(p(u_{i+p,j+q}^n - u_{i,j}^n) / h_1, (v_{i,j} - v_{i-q,j-p}) / h_2 \right) \quad (9)$$

$$(p, q) = (0, 1) : \nabla^{pq} u_{ij}^n = \left((v_{i,j} - v_{i-q,j-p}) / h_1, q(u_{i+p,j+q}^n - u_{i,j}^n) / h_2 \right) \quad (10)$$

The formulas (7)-(8) can be understood as an approximation of the gradient in the point x_{ij}^{pq} . This is also an approximation of the gradient at the center of the edge for the dual mesh.

Because of the fact that the gradients can achieve zero values we use the so-called Evans-Spruck type regularization [4], which can be for our scheme defined as

$$Q_{ij}^{pq;n-1} = \sqrt{\varepsilon^2 + |\nabla^{pq} u_{ij}^{n-1}|^2}, \bar{Q}_{ij}^{n-1} = \frac{1}{4} \sum_{|p|+|q|=1} Q_{ij}^{pq;n-1}$$

as a regularized absolute value of the gradient on pixel sides, and the regularized averaged gradient inside the finite volume, respectively, computed by the solution known from the previous time step $n-1$. For the dual mesh we have:

$$\begin{aligned} Q_{ij}^{1,0;n-1} &= \bar{Q}_{ij}^{0,-1;n-1}, & Q_{ij}^{-1,0;n-1} &= \bar{Q}_{i-1j}^{0,-1;n-1} \\ Q_{ij}^{0,1;n-1} &= \bar{Q}_{ij}^{-1,0;n-1}, & Q_{ij}^{0,-1;n-1} &= \bar{Q}_{ij-1}^{-1,0;n-1} \end{aligned} \quad (11)$$

When we combine all the mentioned considerations we conclude with the following approximation (the same for the dual mesh)

$$\sum_{|p|+|q|=1} \int_{e_{ij}^{pq}} \frac{1}{|\nabla u^{n-1}|} \frac{\partial u^n}{\partial \nu} ds \approx \sum_{|p|+|q|=1} m(e_{ij}^{pq}) \frac{1}{Q_{ij}^{pq;n-1}} \frac{u_{i+p,j+q}^n - u_{ij}^n}{h_{ij}^{pq}} \quad (12)$$

Putting together the right hand sides of (6) and (12) and considering zero Neumann boundary conditions, we can write the following linear system of equations, which has to be solved at every discrete time step $n, n=1, \dots, N$, where N is a total number of filtering steps:

Fully-discrete semi-implicit discrete duality finite volume scheme: Let $u_{ij}^0, v_{ij}^0, i=1, \dots, N_1, j=1, \dots, N_2$ be given discrete initial values for the original

and dual mesh respectively. Then, for $n = 1, \dots, N$ we look for $u_{ij}^n, v_{ij}^n, i = 1, \dots, N_1, j = 1, \dots, N_2$, satisfying

$$\frac{u_{ij}^n m(\overline{V}_{ij})}{\overline{Q}_{ij}^{n-1}} + \tau \sum_{|p|+|q|=1} \frac{(u_{ij}^n - u_{i+p, j+q}^n) m(e_{ij}^{pq})}{\overline{Q}_{ij}^{pq; n-1} h_{ij}^{pq}} = \frac{m(\overline{V}_{ij}) u_{ij}^{n-1}}{\overline{Q}_{ij}^{n-1}} \quad (13)$$

$$\frac{v_{ij}^n m(\overline{V}_{ij})}{\overline{Q}_{ij}^{n-1}} + \tau \sum_{|p|+|q|=1} \frac{(v_{ij}^n - v_{i+p, j+q}^n) m(\overline{e}_{ij}^{pq})}{\overline{Q}_{ij}^{pq; n-1} h_{ij}^{pq}} = \frac{m(\overline{V}_{ij}) v_{ij}^{n-1}}{\overline{Q}_{ij}^{n-1}} \quad (14)$$

However, we restrict our considerations to uniform rectangular co-volumes with size length h . Then, e.g.,

$$m(\overline{V}_{ij}) = h^2, m(e_{ij}^{pq}) = h, h_{ij}^{pq} = h \quad (15)$$

Conclusion

The main objective of this paper was to describe the new scheme used for the numerical solving of curvature driven level set equations. Our next objective is to prove the stability of this method via computational testing.

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