# Fuzzy Interpolation According to Fuzzy and Classical Conditions

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*We will be focused on the interpolation approach to a computation with fuzzy data. A definition of interpolation of fuzzy data, which stems from the classical approach, is proposed. We investigate another approach to fuzzy interpolation (published in [5]) with relaxed interpolation condition. We prove that even if the interpolation condition is relaxed the related algorithm gives an interpolating fuzzy function which fulfils the interpolation condition in the classical sense.*

*Keywords: Fuzzy function; fuzzy equivalence, fuzzy space, fuzzy interpolation, fuzzy rule base interpolation*

## 1 Introduction

In the following, we will deal with a problem of a fuzzy interpolation. We will first recall a classical approach to interpolation, because the problem of fuzzy interpolation closely relates to it.

Let $f$ be a real function of a real argument with a domain $\mathcal{M} = \{x_i,\ i = 1,\ldots,n\} \subset \mathbb{R}$, and $\{(x_i,\ f(x_i)),\ i = 1,\ldots,n\} \subseteq \mathbb{R}$ be the interpolation data. Let $\mathcal{M} \subset P \subseteq \mathbb{R}$. An interpolation function $g : P \longrightarrow \mathbb{R}$ is a function that fulfils the interpolation condition:

$$f(x_i) = g(x_i),\ i = 1,\ldots,n.$$

In this paper, we will be focused on the interpolation approach to a computation with fuzzy data. A precise definition of *interpolation of fuzzy data* will be given below in the subsection 2.4. Freely speaking, this is a problem of extension of a fuzzy function given on a restricted domain to a fuzzy function given on a wider domain (similar to the case considered above).

There are other approaches to the problem of fuzzy interpolation. They differ one from the other one by restrictions on interpolation functions. The following list remembers the most popular approaches : level cuts interpolation [17, 18], analogy-based interpolation [3, 4, 6], interpolation by convex completion [8, 23], interpolation by geometric transformations [1], interpolation in a family of interpolating relations [2], polar cut interpolation [15], interpolation based on closeness relations [5], flank functions interpolation [13, 14], analytic fuzzy relation-based interpolation [20], and fuzzy interpolation based on fuzzy functions [11].

The sections below are arranged as follows : basic concepts as well as definition of fuzzy interpolation will be given in Section 2. In Section 3, we will recall the approach to fuzzy interpolation introduced by Godo, Esteva, ets. in [5]. The last Section 4 is devoted to a new approach.

# 2 Fuzzy Interpolation and Its Analytic Representation

In this section, we will introduce the problem of fuzzy interpolation as a problem of extension of a partially given fuzzy function. Moreover, we expect that a solution will be represented analytically with the help of a structure known as residuated lattice.

First of all, we will recall the notion of residuated lattice, then we will introduce the notion of fuzzy space and fuzzy function. Finally, we will discuss a certain class of interpolating fuzzy functions and their analytical representation.

## 2.1 Residuated Lattice

A residuated lattice is an ordered algebraic structure with two residuated binary operation. We will recall its definition from [19].

**Definition 1**
*A residuated lattice*[*)] *is an algebra*

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle.$$

*with a support $L$ and four binary operations and two constants such that*

- $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice where the ordering $\leq$ defined using operations $\vee, \wedge$ as usual, and $\mathbf{0}, \mathbf{1}$ are the least and the greatest elements, respectively;*
- $\langle L, *, \mathbf{1} \rangle$ *is a commutative monoid, that is, $*$ is a commutative and associative operation with the identity $a * \mathbf{1} = a$;*
- *the operation $\rightarrow$ is a residuation operation with respect to $*$, i.e.,*

$$a * b \leq c \quad \textit{iff} \quad a \leq b \rightarrow c.$$

A residuated lattice is complete if its underlying lattice is complete.

The derived operation is biresiduum:

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

Our investigation will be based on Łukasiewicz algebra $\mathcal{L}_{Ł}$. It is a residuated lattice with the support $L = [0, 1]$ where

$$a * b = 0 \vee (a + b - 1),$$
$$a \rightarrow b = 1 \wedge (1 - a + b),$$
$$a \leftrightarrow b = 1- \mid a - b \mid .$$

The other well known examples of residuated lattice are Boolean algebra, Gödel algebra and product algebra.

## 2.2 $L$-fuzzy Space

Assume that we are given a complete residuated lattice $\mathcal{L}$ and a non-empty universal set $X \subseteq \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. An *L-valued fuzzy set* is a mapping $A : X \rightarrow L$. A *core* of a fuzzy set $A$ is the set $\mathrm{Core}(A) = \{x \in X | \ A(x) = 1\}$. We say that a fuzzy set is *normal* if there exists $x_A \in X : \ A(x_A) = 1$. The class of $L$-valued fuzzy sets of $X$ will be denoted by $L^X$.

Let $A, \ B \in L^X$ be fuzzy sets. A *fuzzy equality* $(A \equiv B)$ is given by the following formula

$$(A \equiv B) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x))^{\dagger)}. \tag{1}$$

[*)] In this paper we assume a residuated lattice to be bounded, commutative and integral.
[†)] for a general definition of fuzzy equality, e.g., [7, 12, 16]

The fuzzy equality determines a degree of coincidence of two fuzzy sets expressed by an element of the residuated lattice.

It is known that

$$(A \equiv B) = 1 \quad \text{iff} \quad \forall x \in X, \quad A(x) = B(x),$$

In this case we will write $A = B$ instead of $A \equiv B$.

**Definition 2**
*The pair $(L^X, \equiv)$ is a fuzzy space on $X$.*

The definition of fuzzy space introduces a basic set of objects together with the basic relation of equality.

**Example 1**
*Let $X = [a, b]$, $\mathcal{L} = \mathcal{L}_{Ł}$. Let us show how the fuzzy equality $\equiv$ is expressed. Assume that $A,\ B \in [0,1]^{[a,b]}$.*

$$A \equiv B = \bigwedge_{x\in[a,b]} (A(x) \leftrightarrow B(x)) = \bigwedge_{x\in[a,b]} (1- \mid A(x) - B(x) \mid) =$$

$$1 - \bigvee_{x\in[a,b]} \mid A(x) - B(x) \mid .$$

*Thus the pair $([0,1]^{[a,b]},\ 1-\bigvee_{x\in[a,b]} \mid A(x)-B(x) \mid)$ is a fuzzy space on $[a, b]$ determined by Łukasiewicz algebra.*

*In the special case where fuzzy sets on $[a, b]$ are continuous mappings, the fuzzy equality $\equiv$ between them can be simplified to*

$$1 - \bigvee_{x\in[a,b]} \mid A(x) - B(x) \mid = 1 - \max_{x\in[a,b]} \mid A(x) - B(x) \mid = 1 - d(A, B),$$

*where $d(A, B)$ is the distance in the metric space of continuous functions.*

## 2.3 Fuzzy Function

We will use the notion of fuzzy function introduced in [20]. According to [20], a fuzzy function is an ordinary mapping between two fuzzy spaces. In more details,

**Definition 3**
*Let $\mathcal{L}$ be a complete residuated lattice and $(L^X, \equiv)$, $(L^Y, \equiv)$ fuzzy spaces on $X$ and $Y$, respectively. A mapping $f : L^X \longrightarrow L^Y$ is a fuzzy function if for every $A, B \in L^X$,*

$$A = B \text{ implies } f(A) = f(B). \tag{2}$$

Let us remark that there are other definitions of fuzzy function in [7, 9, 10, 16] where fuzzy function is defined as a fuzzy set of function or as a special fuzzy relation.

Below, we give an example of a fuzzy function which is reproduced from [21].

**Example 2 (Fuzzy functions determined by a fuzzy relation)**
*Let $(L^X, \equiv)$, $(L^Y, \equiv)$ be fuzzy spaces on $X$ and $Y$ respectively, $R \in L^{X\times Y}$ a fuzzy relation. For every $A \in L^X$, we define the* $\circ$-composition *of $A$ and $R$ by*

$$(A \circ R)(y) = \bigvee_{x\in X} (A(x) * R(x, y)). \tag{3}$$

*Composition (3) determines the fuzzy set $A \circ R$ on $Y$. The corresponding mapping $f_{\circ R} : A \mapsto A \circ R$ is a fuzzy function defined on the whole fuzzy space $L^X$.*

## 2.4 Fuzzy Interpolation

The problem of fuzzy interpolation includes two subproblems: a choice of a set of interpolation functions and an extension of an original fuzzy function.

In other words, let $\{(A_i, B_i),\ i = 1, \ldots, n\}$ be a set of fuzzy data and $A_i \in L^X,\ i = 1, \ldots, n$ are pairwise different fuzzy sets with respect to $=$, $B_i \in L^Y,\ i = 1, \ldots, n$. Let a fuzzy function $f : A_i \to B_i,\ i = 1, \ldots, n$ have the domain $\mathcal{M} = \{A_1, \ldots, A_n\}$, and $P$ be a domain of an interpolation fuzzy function $g$ where $\mathcal{M} \subset P \subseteq L^X$. Let $\mathcal{N} \subseteq \{g \mid g : P \longrightarrow L^Y\}$ be a chosen subset of a fuzzy function for the fuzzy interpolation. Our goal is to find an fuzzy function $g \in \mathcal{N}$ satisfying the interpolation condition

$$g(A_i) = B_i,\ i = 1, \ldots, n. \tag{4}$$

The fuzzy function $g$ is called *an interpolation fuzzy function* for fuzzy data. Also we call the interpolation fuzzy function $g$ an extension of $f$ on the domain $P$.

We can also rewrite the interpolation condition (4) as follows:

$$A = A_i \text{ implies } g(A) = B_i,\ i = 1, \ldots, n. \tag{5}$$

## 2.5 Similarity and Fuzzy Point

*A binary fuzzy relation* $E$ on $X$ is called a similarity on $X$ if for all $x, y, z \in X$, the following properties hold:

1. $E(x, x) = 1$,
2. $E(x, y) = E(y, x)$,
3. $E(x, y) * E(y, z) \le E(x, z)$.

Let $E$ be a similarity on $X$. A fuzzy set $E_t,\ t \in X$, where $E_t(x) = E(t, x)$ for all $x \in X$ is called an *$E$-fuzzy point* of $X$.

# 3 Fuzzy Rule Base Interpolation

In this contribution, we will investigate another approach to fuzzy interpolation, proposed in [5]. It assumes that an original function is expressed by a set of fuzzy IF-THEN rules

$$RB = \{\text{“If } x \text{ is } A_i \text{ then } y \text{ is } B_i\text{”}\}_{i=1,\ldots,n} \tag{6}$$

($A_i$ and $B_i$ are respective fuzzy sets on $X$ and $Y$), and the rules are sparse in the sense that $A_i \cap A_j = \emptyset$, $i \ne j$. The fuzzy interpolation is proposed to be realized in a form of an algorithm which produces a consequence $B$ to an antecedence $A$ ($A$ and $B$ are fuzzy sets too). An interpolating algorithm should respect the following requirement:

“The more the input $A$ is *close to* $A_i$
the more the output $B$ must be *close to* $B_i$.” (7)

In [5], a general interpolating algorithm is proposed. Below, we give its essential details that characterize relations of closeness on both universes and describe the way of computing $B$ according to the requirement (7).

The closeness relations between two fuzzy sets show how much one is similar or is included into the other one. Let $S = \{S_\lambda;\ 0 \leq \lambda \leq\ +\infty\}$, be any nested family of fuzzy similarity relations on $\mathbb{R}$ such that $S_0$ is the crisp equality and $S_{+\infty}$=1. Then

$$\text{close}_\lambda(E, D) = min(I_{S_\lambda}(D|E), I_{S_\lambda}(E|D)), \tag{8}$$

where

$$I_{S_\lambda}(D|E) = inf_{u\in\mathbb{R}}\left\{E(u) \to (S_\lambda \circ D)(u)\right\}. \tag{9}$$

According to [5], the value of (algorithmically defined) interpolating function at point $A$ is fuzzy set $B$ such that

$$B = \text{Interpol}_{RB}(A) = \cap_{R_i\in K(A)} \cap_{\lambda\geq 0} \text{close}_\lambda(A, A_i) \to (S_{f(\lambda)} \circ B_i), \tag{10}$$

where $K(A)$ is a subset of fuzzy rules related to $A$, $\to$ is the residuum of a left-continuous t-norm $*$, and $\circ$ is the max-* composition. Moreover, for each $0 < \lambda \leq +\infty$,

$$f(\lambda) = \inf\left\{\mu \mid \text{close}_\lambda(A_1, A_2) \leq \text{close}_\mu(B_1, B_2)\right\}, \tag{11}$$

where $\mu$ is any parameter which satisfies the inequality $\text{close}_\lambda(A_1, A_2) \leq \text{close}_\mu(B_1, B_2)$. It is proved in [5] that thus proposed algorithm fulfils the requested requirement (7).

It is seen from the description above, that a specification of the algorithm requires a choice of a parametric family of fuzzy similarity relations *S* and operations from a certain residuated lattice. One partial specification was proposed in [5] as well. It is based on an arbitrary left-continuous t-norm $*$ and the parametric family of fuzzy similarity relations on $\mathbb{R}^2$:

$$S_\lambda(x, y) = \max(1 - \frac{\mid x - y \mid}{\lambda}, 0). \tag{12}$$

# 4 Main Result

Our purpose it to show that the interpolation algorithm presented in [5] and based on the fuzzy prescription (7) satisfies the interpolation condition in the sense (4). It means that the interpolation function $\text{Interpol}_{RB}(A)$ (cf. (10)) fulfils the interpolation condition in the form

$$(\forall A_i) \quad \text{Interpol}_{RB}(A_i)\ =\ B_i,\ i = 1, \ldots, n.$$

## 4.1 Assumptions and Preliminaries

The Łukasiewicz algebra is chosen as an underlying residuated lattice. Without lost of generality, we assume that only two IF-THEN fuzzy rules specify an original function so that the subset of IF-THEN fuzzy rules connected with $A$ is

$$K(A) = \{A_1 \to B_1, A_2 \to B_2\}.$$

Let $A_1,\ A_2$ be normal and triangular shaped fuzzy sets (inputs) defined on $X \subset \mathbb{R}$ and $B_1,\ B_2$ be normal and triangular shaped fuzzy sets (outputs) defined on $Y \subset \mathbb{R}$. Obviously, the core of a triangular shaped fuzzy set consists of one element – the *core point*. Denote core points of $A_1,\ A_2$ by $x_{A_1}$ and $x_{A_2}$, and similarly, core points of $B_1,\ B_2$ by $y_{B_1}$ and $y_{B_2}$, respectively. Denote (arbitrary) normal and triangular shaped fuzzy set on $X \subset \mathbb{R}$ by $A$, and its core point by $x_A$. Our aim is to prove that the fuzzy set $B$ given by (10) fulfils (4).

Let $S_\lambda$ be a similarity on $X$ and $\lambda \geq 0$ a fixed real number. The composition between $S_\lambda$ and $A$ is given by

$$(S_\lambda \circ A)(x) = \bigvee_u (S_\lambda(x, u) * A(u)).$$

In the following proposition we will show how the similarity relation $S_\lambda$ affects a fuzzy set.

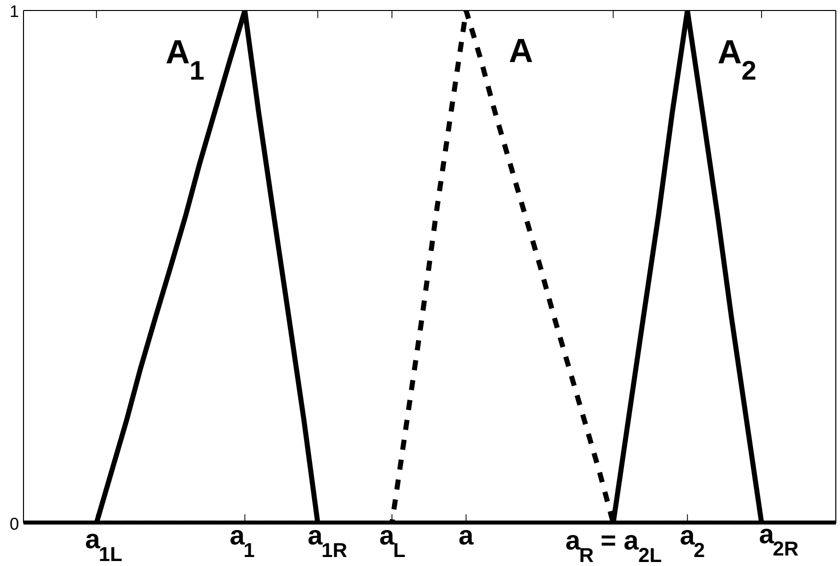


Figure 1: Fuzzy sets on $X$

**Proposition 1**
*Let $A$ be a normal fuzzy set and $x_A \in X$ be its core point. Let moreover, $S^{\lambda}_{x_A}$ be the $S_\lambda$-fuzzy point determined by $x_A$, i.e. $S^{\lambda}_{x_A} = S_\lambda(x_A, x)$. Then for all $x \in X$:*

1. $(S_\lambda \circ A)(x) \geq A(x)$,
2. $(S_\lambda \circ A)(x) \geq S^{\lambda}_{x_A}(x)$,
3. *there exists* $\lambda^* \geq 0$ *such that* $A(x) \leq S^{\lambda^*}_{x_A}(x)$ *and then* $(S_{\lambda^*} \circ A)(x) = S^{\lambda^*}_{x_A}(x)$.

PROOF: By the assumption, $A(x_A) = 1$. We will use properties of a similarity relation and obtain:

1. $(S_\lambda \circ A)(x) = \bigvee_u (S_\lambda(x, u) * A(u)) \geq (S_\lambda(x, x) * A(x)) = A(x)$.
2. $(S_\lambda \circ A)(x) = \bigvee_u (S_\lambda(x, u) * A(u)) \geq (S_\lambda(x, x_A) * A(x_A)) = S_\lambda(x, x_A) = S_\lambda(x, x_A) = S^{\lambda}_{x_A}(x)$.
3. Assume that $A(u) \leq S^{\lambda^*}_{x_A}(u)$ so that the following holds: $(S_{\lambda^*} \circ A)(x) = \bigvee_u (S_{\lambda^*}(x, u) * A(u)) \leq \bigvee_u (S_{\lambda^*}(x, u) * S_{\lambda^*}(x_A, u)) = \bigvee_u (S_{\lambda^*}(x, u) * S_{\lambda^*}(u, x_A)) = S_{\lambda^*}(x, x_A) = S_{\lambda^*}(x_A, x)$. On the other side, $(S_{\lambda^*} \circ A)(x) \geq S^{\lambda^*}_{x_A}(x)$ for $\lambda^* \geq 0$. Therefore, $(S_{\lambda^*} \circ A)(x) = S^{\lambda^*}_{x_A}(x)$.

□

By the proposition above, we can rewrite (8) as follows:

$$\text{close}_\lambda(A_1, A_2) = min(I_{S_{\lambda^*}}(A_2|A_1), I_{S_{\lambda^*}}(A_1|A_2)) = \\ = \bigwedge_x (S_{\lambda^*} \circ A_1 \leftrightarrow S_{\lambda^*} \circ A_2)$$

where

$$I_{S_\lambda}(A_2|A_1) = inf_{x \in \mathbb{R}} \{A_1(u) \to (S_{\lambda^*} \circ A_2)(u)\} = \\ = inf_{x \in \mathbb{R}} \{S_{\lambda^*}(x_{A_1}, x) \to S_{\lambda^*}(x_{A_2}, x)\},$$

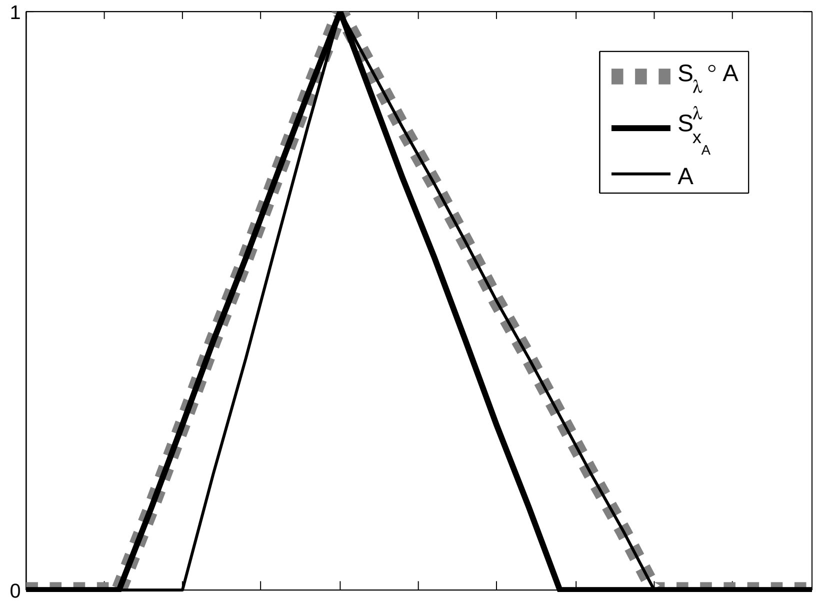


Figure 2: Properties $(S_\lambda \circ A)(x) \geq A(x)$ and $(S_\lambda \circ A)(x) \geq S^\lambda_{x_A}(x)$

and similarly $I_{S_\lambda}(A_1|A_2)$.

Let us simplify the expression (10) by applying assumptions that are accepted at the beginning of this subsection. Moreover, we replace similarities by distances (similar approach has been used in [22]). The distance between two triangular shaped fuzzy sets is considered as a distance between their core points[‡)]:

$$d(A, A_1) = \mid x_A - x_{A_1} \mid,\ d(A, A_2) = \mid x_A - x_{A_2} \mid,$$
$$d(A_1, A_2) = \mid x_{A_1} - x_{A_2} \mid$$

and

$$d(B_1, B_2) = \mid y_{B_1} - y_{B_2} \mid .$$

**Proposition 2**
*Let $A_1,\ A_2$ be normal and triangular shaped fuzzy sets with $x_{A_1},\ x_{A_2} \in X$ as respective cores. Let $S_{\lambda^*}$ be given by (12), and $\lambda^* \geq 0$. Then*

$$\bigwedge_x (S_{\lambda^*} \circ A_1 \leftrightarrow S_{\lambda^*} \circ A_2) = S_{\lambda^*}(x_{A_1}, x_{A_2}).$$

PROOF: We use the following property of the absolute value: $\mid a - b \mid \geq \mid\mid a \mid - \mid b \mid\mid$ or equivalently, $-(\mid\mid a \mid - \mid b \mid\mid) \geq -(\mid a \mid - \mid b \mid)$.

[‡)]Recall that each triangular shaped fuzzy set has exactly one core point so that our definition of a distance is correct.

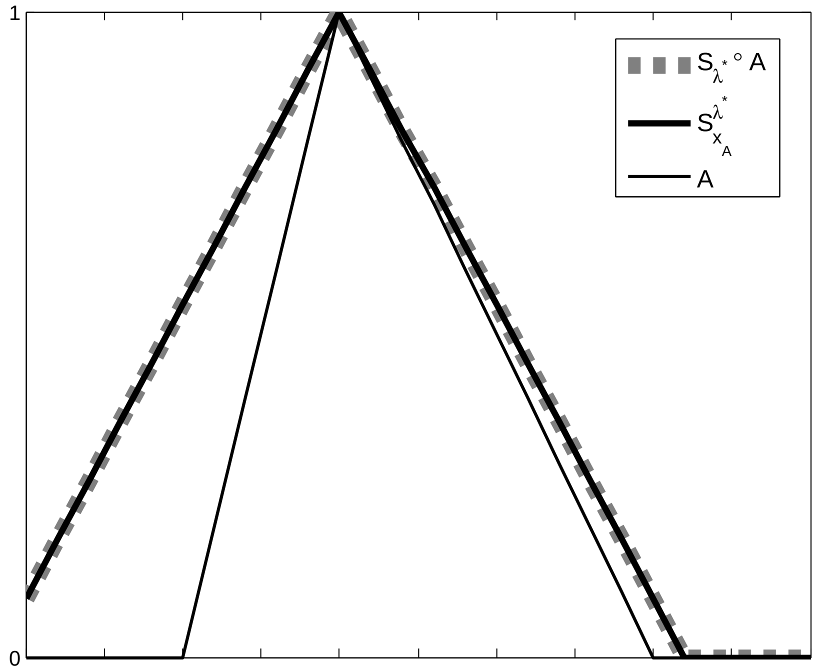


Figure 3: Property $A(x) \le S^{\lambda^*}_{x_A}(x)$, $(S_{\lambda^*} \circ A)(x) = S^{\lambda^*}_{x_A}(x)$

$$\bigwedge_x (S_{\lambda^*} \circ A_1 \leftrightarrow S_{\lambda^*} \circ A_2) = \bigwedge_x (S^{\lambda^*}_{x_{A_1}}(x) \leftrightarrow S^{\lambda^*}_{x_{A_2}}(x)) = \bigwedge_x (1- \mid S_{\lambda^*}(x_{A_1}, x) -$$
$$S_{\lambda^*}(x_{A_2}, x) \mid) = \bigwedge_x \left(1 - \left|1 - \frac{\mid x_{A_1} - x \mid}{\lambda^*} - 1 + \frac{\mid x_{A_2} - x \mid}{\lambda^*}\right|\right) =$$
$$\bigwedge_x \left(1 - \left|\frac{\mid x_{A_2} - x \mid}{\lambda^*} - \frac{\mid x_{A_1} - x \mid}{\lambda^*}\right|\right) = \bigwedge_x \left(1 - \frac{1}{\lambda^*} \mid\mid x_{A_2} - x \mid - \mid x_{A_1} - x \mid\mid\right)$$
$$= 1 - \bigvee_x \frac{1}{\lambda^*}(\mid\mid x_{A_2} - x \mid - \mid x_{A_1} - x \mid\mid) \ge 1 - \bigvee_x \frac{1}{\lambda^*}(|x_{A_2} - x - x_{A_1} + x|) =$$
$$1 - \bigvee_x \frac{1}{\lambda^*}(|x_{A_2} - x_{A_1}|) = 1 - \frac{1}{\lambda^*}(|x_{A_2} - x_{A_1}|) = S_{\lambda^*}(x_{A_1}, x_{A_2})$$

Assume that $x \le x_{A_1} \le x_{A_2}$. Three cases are possible.

1. $x \le x_{A_1} \le x_{A_2}$. In this case, $\mid x_{A_2} - x \mid = x_{A_2} - x$. Similarly for $\mid x_{A_1} - x \mid$.

$$1 - \bigvee_x \frac{1}{\lambda^*}(\mid\mid x_{A_2} - x \mid - \mid x_{A_1} - x \mid\mid) = 1 - \bigvee_x \frac{1}{\lambda^*}(|x_{A_2} - x - x_{A_1} + x|)$$
$$= 1 - \bigvee_x \frac{1}{\lambda^*}(|x_{A_2} - x_{A_1}|) = 1 - \frac{1}{\lambda^*}(|x_{A_2} - x_{A_1}|) = S_{\lambda^*}(x_{A_1}, x_{A_2})$$

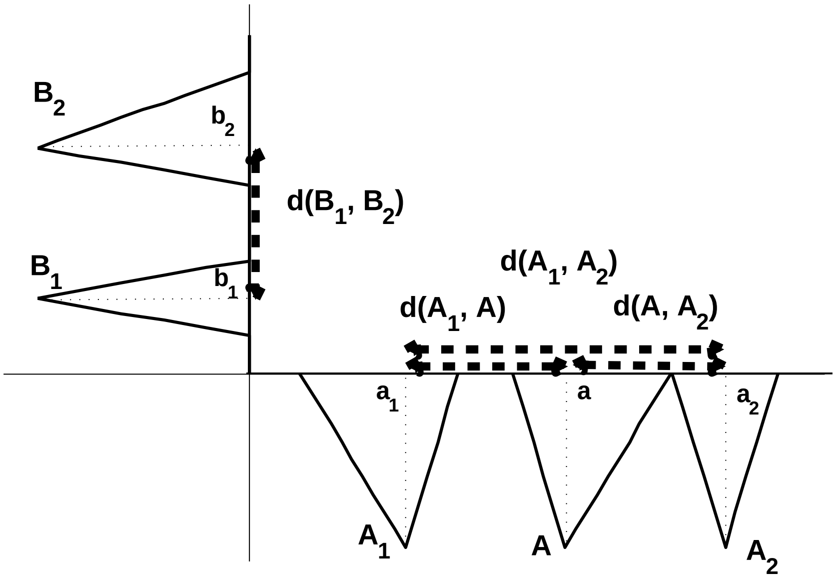


Figure 4: Distances between fuzzy sets

2. Assume that $x_{A_1} \leq x_{A_2} \leq x$ so that $\mid x_{A_2} - x \mid = x - x_{A_2}$, and similarly for $\mid x_{A_1} - x \mid$.

$$1 - \bigvee_x \frac{1}{\lambda^*}(\mid\mid x_{A_2} - x \mid - \mid x_{A_1} - x \mid\mid) = 1 - \bigvee_x \frac{1}{\lambda^*}(|-x_{A_2} + x + x_{A_1} - x|)$$
$$= 1 - \bigvee_x \frac{1}{\lambda^*}(|x_{A_1} - x_{A_2}|) = 1 - \frac{1}{\lambda^*}(|x_{A_2} - x_{A_1}|) = S_{\lambda^*}(x_{A_1}, x_{A_2})$$

3. Finally, let $x_{A_1} \leq x \leq x_{A_2}$ so that$\mid x_{A_2} - x \mid$ is equal to $x_{A_2} - x$. The absolute value $\mid x_{A_1} - x \mid$ is equal to $\mid x_{A_1} - x \mid = x - x_{A_1}$. Without lost of generality, let us choose $x = x_{A_1}$.

$$\bigwedge_x \left(1 - \frac{1}{\lambda^*}(\mid\mid x_{A_2} - x \mid - \mid x_{A_1} - x \mid\mid)\right) =$$
$$\bigwedge_x \left(1 - \frac{1}{\lambda^*}(|x_{A_2} - x + x_{A_1} - x|)\right) \leq 1 - \frac{1}{\lambda^*}(|x_{A_2} - x_{A_1} + x_{A_1} - x_{A_1}|) =$$
$$1 - \frac{1}{\lambda^*}(|x_{A_2} - x_{A_1}|) = 1 - \frac{1}{\lambda^*}(|x_{A_2} - x_{A_1}|) = S_{\lambda^*}(x_{A_1}, x_{A_2})$$

□

## 4.2 The Main Result

The following are assumptions of the main result:

1. Let $\lambda_1 \leq \lambda_2$ then $S_{\lambda_1} \leq S_{\lambda_2}$,

2. $A_1, A_2$ are normal fuzzy sets and $x_{A_1} \leq x_{A_2}$,

3. $\exists \lambda^* \geq 0 : \forall i = 1, 2 \quad A_i \leq S^{\lambda^*}_{x_{A_i}}$,

4. $S^{\lambda^*}_{x_{A_1}} \wedge S^{\lambda^*}_{x_{A_2}} = 0.$

We will describe and concretize the other parts of the expression (10) by means of distances.

Let $\lambda^* \leq \lambda \leq +\infty$. Let us remind that the same parametric family of fuzzy similarity relations on $\mathbb{R}^2$ is given by (12).

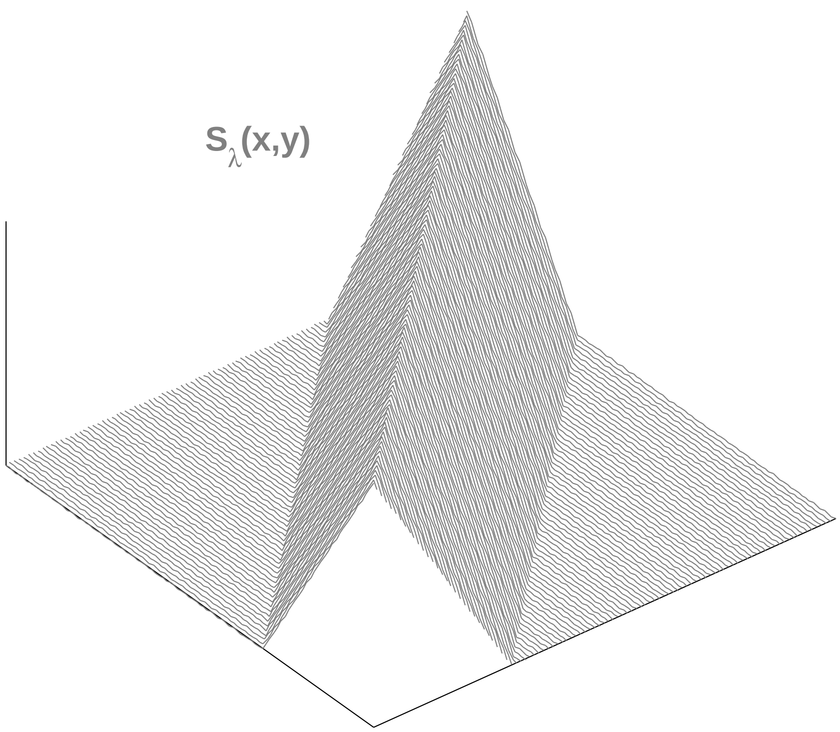


Figure 5: Similarity relation

The idea is that we extend the fuzzy set $A_1$ by applying to it $S_\lambda$ where $\lambda^* \leq \lambda \leq +\infty$.

Now we rewrite the expression (8) (degree of closeness) using distances. For each $0 < \lambda \leq +\infty$,

$$\text{close}_\lambda(A_1, A_2) = min(I_{S_\lambda}(A_2|A_1), I_{S_\lambda}(A_1|A_2)) = \frac{\lambda - \mid x_{A_1} - x_{A_2} \mid}{\lambda}, \tag{13}$$

and respectively,

$$\text{close}_\lambda(A, A_i) = \frac{\lambda - \mid x_A - x_{A_i} \mid}{\lambda}, \quad i = 1, 2. \tag{14}$$

The respective value $f(\lambda)$ can now be expressed as

$$f(\lambda) = \frac{\lambda \mid y_{B_1} - y_{B_2} \mid}{\mid x_{A_1} - x_{A_2} \mid}. \tag{15}$$

The expression (15) is equivalent to (11). However, (15) is represented with the help of distances and by this, its meaning is clearer.

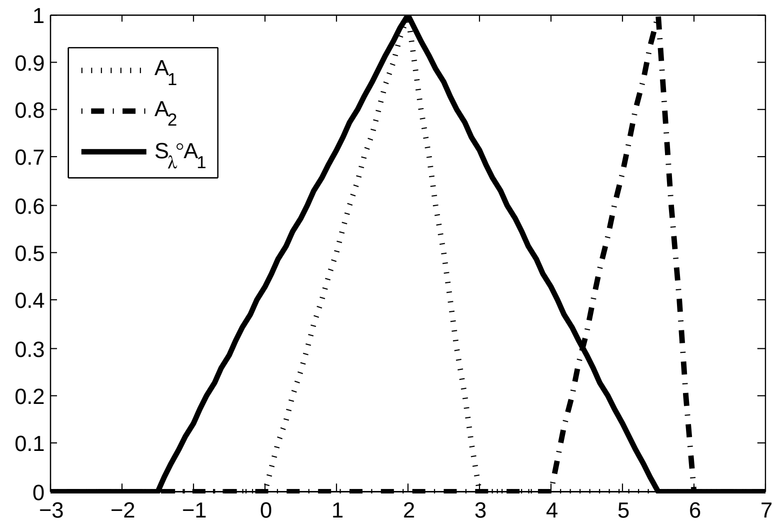


Figure 6: Extension fuzzy set $A_1$

Finally, we will characterize the fuzzy set $B$ given by (10) with the help of distances too.

$$B = \bigwedge_{i=1}^{2} \bigwedge_{\lambda} (\text{close}_\lambda(A, A_i) \to (S_{f(\lambda)} \circ B_i)(y)) =$$

$$\bigwedge_{i=1}^{2} \bigwedge_{\lambda} \left( 1 - 1 + \frac{\mid x_A - x_{A_i} \mid}{\lambda} + (S_{f(\lambda)} \circ B_i)(y) \right) =$$

$$\bigwedge_{i=1}^{2} \bigwedge_{\lambda} \left( \frac{\mid x_A - x_{A_i} \mid}{\lambda} + (S_{f(\lambda)} \circ B_i)(y) \right)$$

Now, we can prove that the interpolation condition (5) is fulfilled.

**Theorem 1**
*If* $A = A_i$ *then* $B = B_i$ *for* $i = 1, 2$.

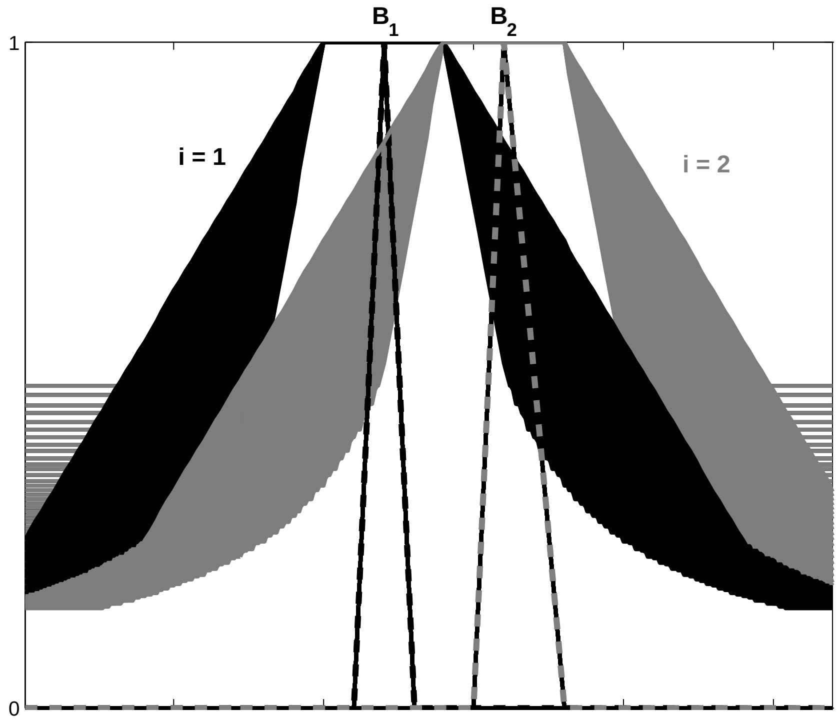


Figure 7: Construction of output fuzzy set $B$

PROOF: Conclusion

$$B = \bigwedge_{i=1}^{2} \bigwedge_{\lambda} (\text{close}_\lambda(A, A_i) \to (S_{f(\lambda)} \circ B_i)(y)) =$$

$$\bigwedge_{i=1}^{2} \bigwedge_{\lambda} \left( 0 \vee \left( 1 - 1 + \frac{\mid x_A - x_{A_i} \mid}{\lambda} + (S_{f(\lambda)} \circ B_i)(y) \right) \right) =$$

$$\bigwedge_{i=1}^{2} \bigwedge_{\lambda} \left( 0 \vee \left( \frac{\mid x_A - x_{A_i} \mid}{\lambda} + (S_{f(\lambda)} \circ B_i)(y) \right) \right) =$$

$$\left[ \bigwedge_{\lambda} \left( \frac{\mid x_A - x_{A_1} \mid}{\lambda} + (S_{f(\lambda)} \circ B_1)(y) \right) \right] \bigwedge$$

$$\left[ \bigwedge_{\lambda} \left( \frac{\mid x_A - x_{A_2} \mid}{\lambda} + (S_{f(\lambda)} \circ B_2)(y) \right) \right] = B' \wedge B''$$

For each $i = 1, 2$, we will prove that $A = A_i \;\Rightarrow\; B = B_i$.

Let $A = A_1$.

$$B' = \bigwedge_{\lambda} (\text{close}_{\lambda}(A_1, A_1) \to (S_{f(\lambda)} \circ B_1)(y)) =$$

$$\bigwedge_{\lambda} \left( \frac{| x_{A_1} - x_{A_1} |}{\lambda} + (S_{f(\lambda)} \circ B_1)(y) \right) =$$

$$\bigwedge_{0<\lambda\leq\lambda'} \left( \frac{| x_{A_1} - x_{A_1} |}{\lambda} + (S_{f(\lambda)} \circ B_1)(y) \right) \wedge$$

$$\bigwedge_{\lambda>\lambda'} \left( \frac{| x_{A_1} - x_{A_1} |}{\lambda} + (S_{f(\lambda)} \circ B_1)(y) \right) =$$

$$\bigwedge_{0<\lambda\leq\lambda'} \big(0 + (S_{f(\lambda)} \circ B_1)(y)\big) \wedge \bigwedge_{\lambda>\lambda'} \big(0 + (S_{f(\lambda)} \circ B_1)(y)\big) =$$

$$B_1(y) \wedge \left[ \bigwedge_{\lambda>\lambda'} \big(0 + (S_{f(\lambda)} \circ B_1)(y)\big) \right] = B_1(y) \wedge \left[ \bigwedge_{\lambda>\lambda'} \big((S_{f(\lambda)}(y_{B_1}, y)\big) \right] = B_1$$

The latter equality follows from $\bigwedge_{0<\lambda\leq\lambda'} \big((S_{f(\lambda)} \circ B_1)(y)\big) = B_1$ which can be justified by the following chain of inequalities:

$$f(\lambda') \leq | y_{B_1} - y |,$$

$$\lambda' \frac{| y_{B_1} - y_{B_2} |}{| x_{A_1} - x_{A_2} |} \leq | y_{B_1} - y |,$$

$$\lambda' \frac{| y_{B_1} - y_{B_2} |}{| x_{A_1} - x_{A_2} |} \leq | y_{B_1} - y |,$$

$$\lambda' \leq | y_{B_1} - y | \frac{| x_{A_1} - x_{A_2} |}{| y_{B_1} - y_{B_2} |}.$$

$$B'' = \bigwedge_{\lambda}(\text{close}_{\lambda}(A_1, A_2) \to (S_{f(\lambda)} \circ B_2)(y)) =$$

$$\bigwedge_{\lambda}\left(\frac{\mid x_{A_1} - x_{A_2} \mid}{\lambda} + (S_{f(\lambda)} \circ B_2)(y)\right) =$$

$$\bigwedge_{\lambda}\left(\frac{\mid x_{A_1} - x_{A_2} \mid}{\lambda} + (1 - \frac{\mid y_{B_2} - y \mid}{f(\lambda)} \vee 0)\right) =$$

$$\bigwedge_{\lambda}\left(1 + \frac{\mid x_{A_1} - x_{A_2} \mid}{\lambda} - \frac{\mid y_{B_2} - y \mid}{f(\lambda)}\right) =$$

$$\bigwedge_{\lambda}\left(1 - \left(\frac{\mid y_{B_2} - y \mid}{f(\lambda)} - \frac{\mid x_{A_1} - x_{A_2} \mid}{\lambda}\right)\right) =$$

$$\bigwedge_{\lambda}\left(1 - \left(\frac{\mid y_{B_2} - y \mid}{\frac{\lambda |y_{B_1} - y_{B_2}|}{|x_{A_1} - x_{A_2}|}} - \frac{\mid x_{A_1} - x_{A_2} \mid}{\lambda}\right)\right) =$$

$$\bigwedge_{\lambda}\left(1 - \left(\frac{\mid y_{B_2} - y \mid\mid x_{A_1} - x_{A_2} \mid}{\lambda \mid y_{B_1} - y_{B_2} \mid} - \frac{\mid x_{A_1} - x_{A_2} \mid}{\lambda}\right)\right) =$$

$$\bigwedge_{\lambda}\left(1 - \frac{1}{\lambda}\left(\frac{\mid y_{B_2} - y \mid\mid x_{A_1} - x_{A_2} \mid}{\mid y_{B_2} - y_{B_1} \mid} - \mid x_{A_1} - x_{A_2} \mid\right)\right) =$$

$$\bigwedge_{\lambda}\left(1 - \frac{1}{\lambda}\left(\left|\frac{(y_{B_2} - y)(x_{A_1} - x_{A_2})}{(y_{B_2} - y_{B_1})}\right| - \mid x_{A_1} - x_{A_2} \mid\right)\right) \geq$$

$$\bigwedge_{\lambda}\left(1 - \frac{1}{\lambda}\left(\left|\frac{(y_{B_2} - y)(x_{A_1} - x_{A_2}) - ((x_{A_1} - x_{A_2})(y_{B_2} - y_{B_1}))}{(y_{B_2} - y_{B_1})}\right|\right)\right) =$$

$$\bigwedge_{\lambda}\left(1 - \frac{1}{\lambda}\left(\left|\frac{-y(x_{A_1} - x_{A_2}) + x_{A_1}y_{B_1} - x_{A_2}y_{B_1}}{(y_{B_2} - y_{B_1})}\right|\right)\right) =$$

$$\bigwedge_{\lambda}\left(1 - \frac{1}{\lambda}\left(\left|\frac{y_{B_1}(x_{A_1} - x_{A_2}) - y(x_{A_1} - x_{A_2})}{(y_{B_2} - y_{B_1})}\right|\right)\right) =$$

$$\bigwedge_{\lambda}\left(1 - \frac{1}{\lambda}\left(\left|\frac{(x_{A_1} - x_{A_2})(y_{B_1} - y)}{(y_{B_2} - y_{B_1})}\right|\right)\right) =$$

$$\bigwedge_{\lambda}\left(1 - \frac{1}{\lambda}\left(\left|\frac{(x_{A_1} - x_{A_2})}{(y_{B_2} - y_{B_1})}\right| \mid y_{B_1} - y \mid\right)\right) =$$

$$\bigwedge_{\lambda}\left(1 - \frac{1}{\lambda}\left|\frac{(x_{A_1} - x_{A_2})}{(y_{B_1} - y_{B_2})}\right| (\mid y_{B_1} - y \mid)\right) =$$

$$\bigwedge_{\lambda}\left(\left(1 - \frac{1}{f(\lambda)}\left(\mid (y_{B_1} - y) \mid\right)\right) \vee 0\right) =$$

$$\bigwedge_{\lambda}\left(S_{f(\lambda)} \circ B_1\right)$$

Finally,

$$A = A_1 \Rightarrow B = B' \wedge B'' = B_1 \wedge B'' = B_1,$$

where $B'' \geq \bigwedge_{\lambda}\left(S_{f(\lambda)} \circ B_1\right)$ and $\bigwedge_{\lambda}\left(S_{f(\lambda)} \circ B_1\right) = B_1$.

Similarly for $A = A_2$.

So it holds

$$A = A_i \Rightarrow B = B_i, \; i = 1, \; 2.$$

□

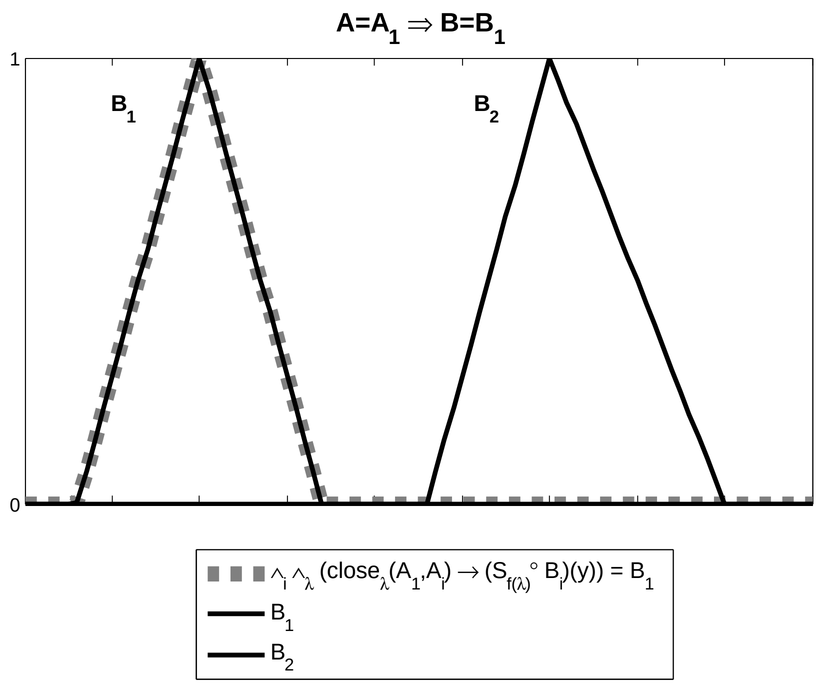


Figure 8: Interpolation condition, $A = A_1 \rightarrow B = B_1$

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## Conclusions

We have proposed a definition of interpolation of fuzzy data, which stems from the classical approach based on rigorous interpolation condition. We investigated another approach to fuzzy interpolation (published in [5]) with relaxed interpolation condition. We simplified and illustrated in various pictures the interpolation algorithm that is based on the proposed in [5] approach. We proved that even if the interpolation condition is relaxed the related algorithm gives an interpolating fuzzy function which fulfils the interpolation condition in the classical sense. Thus the interpolation algorithm in [5] is in the agreement with the definition of interpolation of fuzzy data which we proposed.

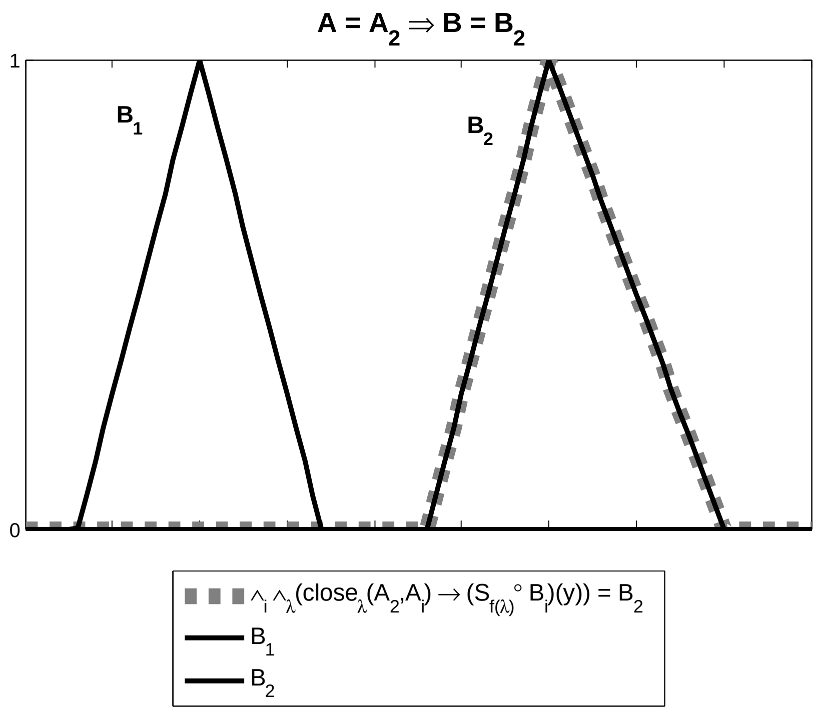


Figure 9: Interpolation condition, $A = A_2 \to B = B_2$

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