

Asymmetric general Choquet integrals

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Abstract: A notion of a generated chain variation of a set function m with values in $[-1, 1]$ is proposed. The space BgV of set functions of bounded g -chain variation is introduced and properties of set functions from BgV are discussed. A general Choquet integral of bounded \mathcal{A} -measurable function is defined with respect to a set function $m \in BgV$. A constructive method for obtaining this asymmetric integral is considered. A general fuzzy integral of bounded g -variation, comonotone \oplus -additive and positive \ominus -homogenous is represented by a general Choquet integral. The representation of a general Choquet integral in terms of a pseudo Lebesgue-Stieltjes integral is obtained.

Keywords: symmetric pseudo-operations, non-monotonic set function, general fuzzy integral, asymmetric Choquet integral

1 Introduction

The Choquet integral is often used in economics, pattern recognition and decision analysis as nonlinear aggregation tool [4, 5, 6, 20, 21, 23, 24]. Most of the studies of non-additive set functions and integrals have been focused to the case when their values are in non-negative interval (fuzzy measures), e.g., $[0, 1]$. A fuzzy measure $m : \mathcal{A} \rightarrow [0, 1]$ (or $[0, \infty]$), $m(\emptyset) = 0$ is a non-decreasing set function, defined on σ -algebra \mathcal{A} . Integrals can be viewed as an extension of underlining measures, see [9, 10].

Choquet integral (introduced in [3]) of \mathcal{A} -measurable non-negative function f with respect to a fuzzy measure $m : \mathcal{A} \rightarrow [0, \infty]$ is defined by

$$C_m(f) = \int_0^\infty m\{x | f(x) \geq t\} dt.$$

The main properties of the Choquet integral are monotonicity and comonotone additivity, see [4, 18]. For a finite fuzzy measure m and \mathcal{A} -measurable $f : X \rightarrow \mathbb{R}$, $f^+ = f \vee 0$,

$f^- = (-f) \vee 0$ we have

$$C_m(f) = C_m(f^+) - C_{\bar{m}}(f^-),$$

where \bar{m} is the conjugate set function of a fuzzy measure m , given by $\bar{m}(E) = m(X) - m(E^c)$, for $E \in \mathcal{A}$, where $E^c = X \setminus E$. The last integral is known under the name *asymmetric Choquet integral*. In [16] it has been shown that this integral is well defined on the class of bounded \mathcal{A} -measurable functions with respect to all real-valued set functions, $m : \mathcal{A} \rightarrow \mathbb{R}$ of bounded chain variation, such that $m(\emptyset) = 0$, even if they are non-monotonic. The asymmetric Choquet integral is linear with respect to m , hence (see [16, 18])

$$C_m(f) = C_{m_1}(f) - C_{m_2}(f).$$

Fuzzy integrals corresponding to an appropriate couple (\oplus, \odot) of pseudo-operations have been studied in [12, 13, 17, 18, 19, 25]. Symmetric pseudo-operations are introduced in [6, 7]. A construction of general fuzzy integral has been studied in [2, 10, 25]. As a special type of such integral, the Choquet-like integral, introduced in [12], is defined with respect to pseudo-operations with a generator. It can be viewed as a transformation of the Choquet integral. The Choquet-like integral related to some non-decreasing function $g : [0, 1] \rightarrow [0, \infty]$, $g(0) = 0$, defined for a non-negative \mathcal{A} -measurable function f and a fuzzy measure m , is given by

$$C_m^g(f) = g^{-1}(C_{g \circ m}(g \circ f)) \quad (1)$$

This integral is also defined for a real-valued function f , if for g is taken its odd extension to the whole real line [12, 13], and we shall call it a general Choquet integral.

The aim of this paper is to present a general Choquet integral defined with respect to set functions of bounded g -chain variation. As we shall see, this integral is of bounded g -variation asymmetric, comonotone \oplus -additive and positively \odot -homogenous.

The paper is organized as follows. Section 2 is devoted to preliminary notions and definitions of symmetric pseudo-operations. In Section 3 we introduce a g -chain variation of set functions and we consider the space of set functions of bounded g -chain variation BgV . In Section 4 we introduce the notion of a signed \oplus_s -measure. A pseudo-difference representation of a signed \oplus_s -measure is obtained. In Section 5 we introduce a general fuzzy integral defined with respect to $m \in BgV$. We consider its relation with the asymmetric general Choquet integral, i.e., Choquet-like integral (defined by (1), w.r.t. $m \in BgV$) and present its representation in the term of a pseudo Lebesque-Stiltjes integral. As a consequence, in the case of an underlining signed \oplus_s -measure this integral reduces to a pseudo Lebesque integral.

2 Symmetric pseudo-operations

We recall definitions of a t-conorm and pseudo-operations according to [6, 7, 9, 10].

Definition 1 A triangular conorm (t-conorm) is a commutative, associative, non-decreasing function $S : [0, 1]^2 \rightarrow [0, 1]$, with neutral element 0.

Definition 2 An additive generator $s : [0, 1] \rightarrow [0, \infty]$ of a t -conorm S (if it exists) is left continuous at 1, increasing function, such that $s(0) = 0$, and for all $(x, y) \in [0, 1]^2$ we have

$$\begin{aligned} S(x, y) &= s^{(-1)}(s(x) + s(y)), \\ s(x) + s(y) &\in \text{Ran}(s) \cup [s(1), \infty], \end{aligned}$$

where $s^{(-1)}$ is a pseudo-inverse function of s (see[9]).

Definition 3 Let $S : [0, 1]^2 \rightarrow [0, 1]$ be a continuous triangular conorm. Pseudo-addition $\oplus_S : [-1, 1]^2 \rightarrow [-1, 1]$, is defined by

$$x \oplus_S y = \begin{cases} S(x, y), & (x, y) \in [0, 1]^2 \\ -S(|x|, |y|), & (x, y) \in [-1, 0]^2 \\ a, & (x, y) \in [0, 1] \times [-1, 0], x \geq |y| \\ b, & (x, y) \in [0, 1] \times [-1, 0], x \leq |y| \\ 1 \text{ or } -1, & (x, y) \in \{(1, -1), (-1, 1)\} \\ y \oplus_S x, & \text{else,} \end{cases}$$

where $a = \inf\{z \mid S(-y, z) \geq x\}$ and $b = -\inf\{z \mid S(x, z) \geq -y\}$.

The binary operation \oplus_S is commutative, monotone, with neutral element 0. If it is associative, e.g., if S is a strict t -conorm, \oplus_S can be extended to n -ary operation. Then for all n -tuple $(x_1, x_2, \dots, x_n) \in [-1, 1]^n$ we define:

$$\bigoplus_{i=1}^n x_i = \left(\bigoplus_{i=1}^{n-1} x_i \right) \oplus_S x_n. \quad (2)$$

Definition 4 Let S be a continuous t -conorm. The pseudo-difference associated to t -conorm S is given by:

$$x \ominus_S y = x \oplus_S (-y) \quad (3)$$

for all $(x, y) \in [-1, 1]^2 \setminus \{(1, 1), (-1, -1)\}$. By the convention $1 \ominus_S 1 = a$, $a \in \{\pm 1, 0\}$.

Example 1 For all $(x, y) \in [-1, 1]^2 \setminus \{(1, 1), (-1, -1)\}$ and for maximum \vee , Yager t -conorm S_p^Y and Hamacher t -conorm (Einstein sum) S_2^H (see [10]), we have, respectively:

(i) $x \ominus_{\vee} y = \text{sign}(x - y)(|x| \vee |y|)$;

(ii) For $p = 2k - 1$,

$$x \ominus_{S_p^Y} y = \begin{cases} -1, & x^p - y^p < -1, \\ \sqrt[p]{x^p - y^p}, & -1 \leq x^p - y^p \leq 1, \\ 1, & x^p - y^p > 1; \end{cases}$$

(iii) $x \ominus_{S_2^H} y = \frac{x-y}{1-xy}$.

Let S be a strict t -conorm with an additive generator $s : [0, 1] \rightarrow [0, \infty]$. Let $g : [-1, 1] \rightarrow [-\infty, \infty]$ be defined by:

$$g(x) = \begin{cases} s(x), & x \geq 0 \\ -s(-x), & x < 0 \end{cases} . \quad (4)$$

The function g is the symmetric extension of s , so it is a strictly increasing function.

A pseudo-addition \oplus_s can be transformed to a binary operation U on $[0, 1]$, i.e., to a generated uninorm. The results contained in the following proposition have been shown in [6, 7, 9].

Proposition 1 *Let S be a strict t -conorm with an additive generator s , pseudo-addition \oplus_s and function g defined by (4), then:*

(i) for all $x, y \in [0, 1]$

$$x \ominus_s y = g^{-1}(g(x) - g(y));$$

(ii) for all $x, y \in [-1, 1]$

$$x \oplus_s y = g^{-1}(g(x) + g(y)); \quad (5)$$

(iii) for all $z, t \in [0, 1]$

$$U(z, t) = u^{-1}(u(z) + u(t)),$$

where $u : [0, 1] \rightarrow [-\infty, \infty]$, is given by $u(x) = g(2x - 1)$, with the convention $\infty - \infty \in \{\infty, -\infty\}$.

It is clear that (i) holds for all $(x, y) \in [-1, 1]^2 \setminus \{(1, 1), (-1, -1)\}$. It is shown in [7] that $(]-1, 1[, \oplus_s)$ is an Abelian group.

It is a well known fact that a pseudo-multiplication $\odot : [-1, 1]^2 \rightarrow [-1, 1]$, which is distributive with respect to \oplus_s , can be defined using the additive generator of pseudo-addition \oplus_s , i.e., for $g : [-1, 1] \rightarrow [-\infty, \infty]$, \odot is defined by:

$$x \odot y = g^{-1}(g(x)g(y)), \quad (6)$$

for all $(x, y) \in]-1, 1[^2$. The pseudo-multiplication defined in this manner is commutative, associative with neutral element $e_\odot \in]0, 1[$ and distributive with respect to pseudo-addition \oplus_s .

Example 2 *Let \oplus_{S_p} be the pseudo-addition induced by the probabilistic sum $S_p : [0, 1]^n \rightarrow [0, 1]$, defined by*

$$S_p(x_1, x_2, \dots, x_n) = 1 - \prod_{i=1}^n (1 - x_i).$$

The additive generator g of \oplus_{S_p} is defined by:

$$g(x) = \begin{cases} -\ln(1-x), & x \geq 0 \\ \ln(1+x), & x < 0 \end{cases} .$$

Let \odot be given by: $x \odot y = g^{-1}(g(x)g(y))$, for all $x, y \in]-1, 1[$, i.e.,

$$x \odot y = \text{sign}(x \cdot y) \left(1 - e^{-\ln(1-|x|)\ln(1-|y|)} \right).$$

For all $x \in]-1, 1[\setminus\{0\}$ we have:

$$x \odot e_{\odot} = x \quad i \quad x \odot x^{-1} = e_{\odot},$$

where the neutral element is given by $e_{\odot} = 1 - \frac{1}{e}$, and an inverse element, for $x \in]-1, 1[\setminus\{0\}$ is given by $x^{-1} = \text{sign}(x) \left(1 - e^{\frac{1}{\ln(1-|x|)}} \right)$. Hence, $(]-1, 1[\setminus\{0\}, \odot)$ is an Abelian group.

The following result was shown in [15].

Proposition 2 *Let S be a strict t -conorm, pseudo-addition \oplus_s with the generating function g given by (4), and pseudo-multiplication \odot is defined by (6). Then we have:*

- (i) $(]-1, 1[\oplus_s, \odot)$ is a field isomorphic to $(\mathbb{R}, +, \cdot)$
- (ii) The pseudo-multiplication has the next form

$$x \odot y = \text{sign}(x \cdot y) U^{\odot}(|x|, |y|),$$

where the uninorm $U^{\odot} : [0, 1]^2 \rightarrow [0, 1]$ is defined by $U^{\odot}(x, y) = s^{-1}(s(x)s(y))$ for all $x, y \in [0, 1]$, with the convention:

- (a) in the case $\infty \cdot 0 = 0$, U^{\odot} is conjunctive,
- (b) in the case $\infty \cdot 0 = \infty$, U^{\odot} is a disjunctive uninorm.

It is clear now, that the couple of symmetric pseudo-operations (\oplus_s, \odot) can be expressed in terms of a couple of uninorms, or as it is usual by (5) and (6).

3 Space BgV

According to [16, 18], the chain variation of a real valued set function $m : \mathcal{A} \rightarrow \mathbb{R}$, $m(\emptyset) = 0$, for all $E \in \mathcal{A}$, is defined by

$$|m|(E) = \sup \left\{ \sum_{i=1}^n |m(E_i) - m(E_{i-1})| \mid \emptyset = E_0 \subset \dots \subset E_n = E, \quad E_i \in \mathcal{A}, i = 1, \dots, n \right\},$$

where supremum is taken with respect to all finite chains from \emptyset to E . The chain variation $|m|$ of a real-valued set function m is positive, monotone, set function, $|m|(\emptyset) = 0$ and $|m(E)| \leq |m|(E)$ for all $E \in \mathcal{A}$. We say that a real-valued set function m , $m(\emptyset) = 0$, is of bounded chain variation if $|m|(X) < \infty$, and we denote by BV the set of all set functions with the bounded chain variation, vanishing at the empty set. We refer [1, 16, 18] for an exhaustive overview of properties and results related to BV . It is proven in [1, 18] that a real-valued set function m belongs to BV if it can be represented as difference of two monotone set functions v_1 and v_2 .

Definition 5 [15] For a given function $g : [-1, 1] \rightarrow [-\infty, \infty]$, defined by (4), g -chain variation $|m|_g$ of a set function $m : \mathcal{A} \rightarrow]-1, 1[$, $m(\emptyset) = 0$, is defined by

$$|m|_g(E) = g^{-1} \left(\sup \left\{ \sum_{i=1}^n |g(m(E_i)) - g(m(E_{i-1}))| \mid \emptyset = E_0 \subset \dots \subset E_n = E, E_i \in \mathcal{A}, i = 1, \dots, n \right\} \right),$$

for all $E \in \mathcal{A}$ and supremum is taken with respect to all finite chains.

Using the fact that g is an odd function, we easily obtain the following result.

Proposition 3 Let $m : \mathcal{A} \rightarrow]-1, 1[$ be a set function, $m(\emptyset) = 0$, then g -chain variation has the following properties:

- (i) $0 \leq |m|_g(E) \leq 1$, $E \in \mathcal{A}$.
- (ii) $|m|_g(\emptyset) = 0$.
- (iii) $|m(E)| \leq |m|_g(E)$, $E \in \mathcal{A}$.
- (iv) $|m|_g$ is a monotone set function, i.e.,

$$|m|_g(E) \leq |m|_g(F),$$

for all $E \subset F$, $E, F \in \mathcal{A}$.

iv) If $m : \mathcal{A} \rightarrow [0, 1]$ is a monotone set function, then

$$|m|_g(E) = m(E) \quad \text{for all } E \in \mathcal{A}.$$

We say that a set function $m : \mathcal{A} \rightarrow]-1, 1[$, $m(\emptyset) = 0$, is of bounded g -chain variation if $|m|_g(X) < 1$, and we denote by BgV the family of such set functions.

Proposition 4 Let $m_1, m_2 \in BgV$. Then

$$|m_1 \oplus_s m_2|_g(X) \leq |m_1|_g(X) \oplus_s |m_2|_g(X).$$

PROOF: We will use the next notation

$$L = \{\emptyset = E_0 \subset E_1 \subset \dots \subset E_n = F, \quad E_i \in \mathcal{A}, i = 1, \dots, n\}.$$

We denote by C_F all finite chains from \emptyset to F . We have

$$\begin{aligned} |m_1 \oplus_s m_2|_g(X) &= g^{-1} \left(\sup_{L \in C_X} \left\{ \sum_{i=1}^n |g((m_1 \oplus_s m_2)(E_i)) - g((m_1 \oplus_s m_2)(E_{i-1}))| \right\} \right) \\ &= g^{-1} \left(\sup_{L \in C_X} \left\{ \sum_{i=1}^n |g \circ m_1(E_i) + g \circ m_2(E_i)| \right\} \right) \end{aligned}$$

$$\begin{aligned}
& - |g \circ m_1(E_{i-1}) - g \circ m_2(E_{i-1})| \Big\} \Big) \\
& \leq g^{-1} \left(\sup_{L \in \mathcal{C}_X} \left\{ \sum_{i=1}^n |g \circ m_1(E_i) - g \circ m_1(E_{i-1})| \right. \right. \\
& \quad \left. \left. + \sum_{i=1}^n |g \circ m_2(E_i) - g \circ m_2(E_{i-1})| \right\} \right) \\
& \leq g^{-1} \left(g \left(g^{-1} \left(\sup_{L \in \mathcal{C}_X} \left\{ \sum_{i=1}^n |g \circ m_1(E_i) - g \circ m_1(E_{i-1})| \right\} \right) \right) \right) \\
& \quad + g \left(g^{-1} \left(\sup_{L \in \mathcal{C}_X} \left\{ \sum_{i=1}^n |g \circ m_2(E_i) - g \circ m_2(E_{i-1})| \right\} \right) \right) \\
& = |m_1|_g(X) \oplus_S |m_2|_g(X).
\end{aligned}$$

□

Proposition 5 [15] *A set function $m : \mathcal{A} \rightarrow]-1, 1[$, $m(\emptyset) = 0$, belongs to BgV if and only if it can be represented as follows*

$$m = m_1 \ominus_S m_2,$$

where $m_1, m_2 : \mathcal{A} \rightarrow [0, 1]$ are two fuzzy measures.

Proof: We have that $m \in BgV$ if and only if $g \circ m \in BV$. By Theorem 3.10. from [18], there exist two fuzzy measures \tilde{m}_1 and \tilde{m}_2 such that $g \circ m = \tilde{m}_1 - \tilde{m}_2$. Taking $m_1 = g^{-1} \circ \tilde{m}_1$ and $m_2 = g^{-1} \circ \tilde{m}_2$ we obtain the claim. □

4 Signed \oplus_S -measures

In this section we consider a set function $m : \mathcal{A} \rightarrow [-1, 1]$. We will define σ - \oplus_S -additivity of a set function m in the following manner. Let S be a strict t-conorm and \oplus_S a pseudo-addition with an additive generator $g : [-1, 1] \rightarrow [-\infty, \infty]$. First, we define the notion of a convergent \oplus_S -series $\bigoplus_{i=1}^{\infty} a_i$. We have the following situations:

(i) An expression $\bigoplus_{i=1}^{\infty} a_i$ is unambiguously defined if $a_i \geq 0$ for all $i = 1, 2, \dots$. Then

$\{\bigoplus_{i=1}^n a_i\}_{n \in \mathbb{N}}$ is a monotone increasing sequence of reals from the interval $[0, 1]$, hence

$$\bigoplus_{i=1}^{\infty} a_i := \lim_{n \rightarrow \infty} \bigoplus_{i=1}^n a_i, \quad (7)$$

i.e., the sum of \oplus_S -series is equal to a number from the interval $[0, 1[$ and we say that \oplus_S -series is convergent, otherwise it diverges to 1.

(ii) In the case when $a_i \leq 0$, for all $i = 1, 2, \dots$ we have the similar situation as in (i), i.e., the sum of \oplus_S -series is a number from the interval $] -1, 0]$, otherwise it diverges to

-1.

(iii) For $a_i \in [-1, 1]$, $i = 1, 2, \dots$, analogously as in the previous situations, we take (7), i.e., the classical limit value of the sequence of reals $\{\bigoplus_{i=1}^n a_i\}_{n \in \mathbb{N}}$, if it exists, i.e., if it is a number from the interval $] -1, 1[$.

We introduce the notion of $\sigma\text{-}\bigoplus_S$ -additivity as follows. A distorted signed measure μ transformed by g^{-1} , i.e., any real valued signed fuzzy measure $m = g^{-1} \circ \mu$ is $\sigma\text{-}\bigoplus_S$ -additive, if g is an additive generator of pseudo-addition \bigoplus_S and $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$ is an arbitrary signed measure.

Definition 6 A set function $m : \mathcal{A} \rightarrow [-1, 1]$ is a signed \bigoplus_S -measure if there exists a signed measure $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$ (μ assumes at most one of the values from $\{+\infty, \infty\}$) such that:

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = g^{-1}\left(\sum_{i=1}^{\infty} \mu(E_i)\right)$$

is fulfilled for any sequence $\{E_i\}_{i \in \mathbb{N}}$, $E_i \in \mathcal{A}$, satisfying $E_k \cap E_j = \emptyset$ for $k \neq j$, where the series on the right side is either convergent or divergent to $+\infty$ or $-\infty$.

Obviously, we have $m(\emptyset) = 0$ and m takes on at most one of the values from $\{-1, 1\}$.

Proposition 6 Let $m : \mathcal{A} \rightarrow [-1, 1]$ be a signed \bigoplus_S -measure. Then there exist unique fuzzy measures m_1 and m_2 such that

$$m = m_1 \ominus_S m_2.$$

Proof. According to the classical Jordan's theorem of representation of a signed measure (see [8]), we have $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are measures. By Definition 6, for all $E \in \mathcal{A}$ we have

$$\begin{aligned} m(E) &= g^{-1}(\mu(E)) \\ &= g^{-1}(\mu^+(E) - \mu^-(E)) \\ &= g^{-1}(g(g^{-1} \circ \mu^+(E)) - g(g^{-1} \circ \mu^-(E))) \\ &= m_1(E) \ominus_S m_2(E). \end{aligned}$$

□

Example 3 Let $\mu : \mathcal{A} \rightarrow [-\infty, \infty]$ be a signed measure and let m be a set function defined on σ -algebra \mathcal{A} , $m : \mathcal{A} \rightarrow [-1, 1]$ as follows:

$$m(E) = \text{sign}(\mu(E)) \left(1 - e^{-|\mu(E)|}\right).$$

The set function m is a signed \bigoplus_{Sp} -measure.

Remark 1 Let $m : \mathcal{A} \rightarrow [-1, 1]$ be a set function such that $m \in \text{BgV}$. Then there exist m_1 and m_2 such that $m = m_1 \ominus_S m_2$. If the fuzzy measures m_1 and m_2 are S -measures, then m is a signed \bigoplus_S -measure.

5 A general Choquet integral

Let (X, \mathcal{A}) be a measurable space, and \mathcal{F}^+ and \mathcal{F} classes of \mathcal{A} -measurable functions given by

$$\mathcal{F}^+ = \{f \mid f : X \rightarrow [0, 1], \sup_{x \in X} f(x) < 1\},$$

$$\mathcal{F} = \{f \mid f : X \rightarrow [-1, 1], \sup_{x \in X} |f(x)| < 1\},$$

Let the operation \ominus be given by Definition 4. For a set function $m : \mathcal{A} \rightarrow]-1, 1[$, $m(\emptyset) = 0$, we define a pseudo conjugate set function $\bar{m}^\ominus : \mathcal{A} \rightarrow]-1, 1[$ by:

$$\bar{m}^\ominus(E) = m(X) \ominus m(E^c),$$

for all $E \in \mathcal{A}$, where $E^c = X \setminus E$.

Proposition 7 [15] *We have*

- (i) $f = f^+ \ominus f^-$, for any $f \in \mathcal{F}$, where $f^+, f^- \in \mathcal{F}^+$, $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.
- (ii) m is monotone if and only if \bar{m}^\ominus is monotone.
- (iii) Let $m_1, m_2 : \mathcal{A} \rightarrow]-1, 1[$ such that $m_1(X) = m_2(X)$. Then

$$m_1 \leq m_2 \text{ if and only if } \bar{m}_1^\ominus \geq \bar{m}_2^\ominus.$$

In the sequel, \oplus and \odot will denote associative pseudo-operations, defined by (5) and (6), respectively, and \ominus the corresponding pseudo-difference. The measurable functions f and h on X are called *comonotone* [4] if they are measurable with respect to the same chain \mathcal{C} in \mathcal{A} . Equivalently, comonotonicity of functions f and h can be expressed as follows: $f(x) < f(x_1) \Rightarrow h(x) \leq h(x_1)$ for all $x, x_1 \in X$.

Definition 7 Let $\mathbf{I} : \mathcal{F} \rightarrow]-1, 1[$ be a functional. We say that

- (i) \mathbf{I} is monotone if for all $f, h \in \mathcal{F}$

$$f \leq h \Rightarrow \mathbf{I}(f) \leq \mathbf{I}(h),$$

- (ii) \mathbf{I} is comonotone \oplus -additive if

$$\mathbf{I}(f \oplus h) = \mathbf{I}(f) \oplus \mathbf{I}(h)$$

for all comonotone f and h from \mathcal{F} ,

- (iii) \mathbf{I} is positively \odot -homogenous if

$$\mathbf{I}(a \odot f) = a \odot \mathbf{I}(f)$$

for all $a \in [0, 1[$, $f \in \mathcal{F}$,

(iv) \mathbf{I} is of bounded g -variation if $G(\mathbf{I}) < 1$, where a g -variation $G(\mathbf{I})$ of \mathbf{I} is defined by

$$G(\mathbf{I}) = g^{-1} \left(\sup \left\{ \sum_{i=1}^n |g(\mathbf{I}(h_i)) - g(\mathbf{I}(h_{i-1}))| \mid 0 = h_0 \leq \dots \leq h_n = e\mathbf{1}_X, \quad h_i \in \mathcal{F} \right\} \right).$$

Remark 2 Obviously, if $\mathbf{I}: \mathcal{F} \rightarrow]-1, 1[$ is a monotone functional, then g -variation of \mathbf{I} is given by $G(\mathbf{I}) = \mathbf{I}(e\mathbf{1}_X)$.

Let $m \in BgV$ and let $s \in \mathcal{F}$ be a simple function with $Ran(s) = \{s_1, s_2, \dots, s_n\}$. We define

$$\mathbf{I}_m(s) = s_1 \odot m(E_1) \oplus \bigoplus_{i=2}^n (s_i \ominus s_{i-1}) \odot m(E_i), \quad (8)$$

where $-1 < s_1 \leq s_2 \leq \dots \leq s_n < 1$ and $E_i = \{x \in X \mid s(x) \geq s_i\}$.

Proposition 8 [15] Let \mathbf{I}_m be defined by (8). For all simple functions from \mathcal{F} , and for all $m \in BgV$ we have:

(i) \mathbf{I}_m satisfies the properties (ii) and (iii) given in Definition 7.

(ii) $\mathbf{I}_m(s) = \mathbf{I}_m(s^+) \ominus \mathbf{I}_{\bar{m}^\ominus}(s^-)$.

(iii) $\mathbf{I}_m(s) = \mathbf{I}_{m_1}(s) \ominus \mathbf{I}_{m_2}(s)$, where m_1 and m_2 are given by Proposition 5.

(iv) $\mathbf{I}_m(a \cdot \mathbf{1}_E) = \begin{cases} a \odot m(E) & a \in [0, 1[\\ a \odot \bar{m}^\ominus(E) & a \in]-1, 0[\end{cases}$.

We consider now a general fuzzy integral. First we define a general fuzzy integral with respect to a monotone, non-negative function $m \in BgV$ and then with respect to an arbitrary m from BgV .

Definition 8 A general fuzzy integral $\mathbf{I}_m: \mathcal{F} \rightarrow]-1, 1[$ is defined by:

(i) For a fuzzy measure m from BgV

$$\mathbf{I}_m(f) = \sup_{s \in \mathcal{F}^+, s \leq f^+} \mathbf{I}_m(s) \oplus \inf_{-s' \in \mathcal{F}^+, -s' \leq f^-} \mathbf{I}_m(s'). \quad (9)$$

(ii) For $m \in BgV$

$$\mathbf{I}_m(f) = \mathbf{I}_{m_1}(f) \ominus \mathbf{I}_{m_2}(f), \quad (10)$$

where m_1 and m_2 are given by Proposition 5.

A general fuzzy integral $\mathbf{I}_m: \mathcal{F} \rightarrow]-1, 1[$ with respect to a fuzzy measure is monotone. \mathbf{I}_m is asymmetric, i.e.,

$$\mathbf{I}_m(-f) = -\mathbf{I}_{\bar{m}^\ominus}(f),$$

for all $f \in \mathcal{F}$.

Proposition 9 Let $\mathbf{I}_m : \mathcal{F} \rightarrow]-1, 1[$ be a general fuzzy integral with respect to $m \in BgV$. We have:

- (i) \mathbf{I}_m is of bounded g -variation.
- (ii) \mathbf{I}_m satisfies the properties (ii) and (iii) given in Definition 7.
- (iii) $\mathbf{I}_m(f) = \mathbf{I}_m(f^+) \ominus \mathbf{I}_{\bar{m} \circledast}(f^-)$, for all $f \in \mathcal{F}$.

Proof. (i) Let $m \in BgV$, by Proposition 5, $m = m_1 \ominus m_2$, where m_1 and m_2 are fuzzy measures from BgV . $\mathbf{I}_{m_1}, \mathbf{I}_{m_2} : \mathcal{F} \rightarrow]-1, 1[$ are monotone functionals. By definition of g -variation we have $G(-\mathbf{I}) = G(\mathbf{I})$ and

$$G(\mathbf{I}_m) = G(\mathbf{I}_{m_1} \ominus \mathbf{I}_{m_2}) \leq G(\mathbf{I}_{m_1}) \oplus G(\mathbf{I}_{m_2}) = \mathbf{I}_{m_1}(e\mathbf{1}_X) \oplus \mathbf{I}_{m_2}(e\mathbf{1}_X) = m_1(X) \oplus m_2(X) < 1.$$

We obtain (ii) and (iii) by (8), (9), (10) and Proposition 8. \square

Based on the above consideration and results proven in [2, 4, 15, 16, 18] we have the next propositions.

Proposition 10 Let $\mathbf{I}_m : \mathcal{F} \rightarrow]-1, 1[$ be a general fuzzy integral with respect to $m \in BgV$. Then

$$\mathbf{I}_m(f) = C_m^g(f) = g^{-1}(C_{g \circ m}(g \circ f)),$$

where C_m^g is a general Choquet integral.

Proposition 11 Let $\mathbf{I}_m : \mathcal{F} \rightarrow]-1, 1[$ be a general fuzzy integral w.r.t. $m \in BgV$. Then

$$\mathbf{I}_m(f) = g^{-1} \left(LS \int_{[-\infty, \infty]} g(t) d(g \circ F)(t) \right),$$

where the integral on the right-hand side is a pseudo Lebesgue-Stieltjes integral.

Proof. Let $F : [-1, 1] \rightarrow [-1, 1]$ be a function of bounded totally g -variation, i.e.,

$$g^{-1} \left(\sup \left\{ \sum_{i=1}^n |g(F(t_i)) - g(F(t_{i-1}))| \mid -1 \leq t_1 \leq \dots \leq t_n \leq 1, i = 1, \dots, n \right\} \right) < 1. \quad (11)$$

Then there exist two non-decreasing functions F^+ and F^- such that $F = F^+ \ominus F^-$ and a signed \oplus -measure on a σ -algebra of Borel subsets of $[-1, 1]$, induced by F .

Let \mathbf{I}_m be a general fuzzy integral with respect to $m \in BgV$. For $f \in \mathcal{F}$, let F be defined by

$$F(t) = -m\{x \in X \mid f(x) \geq t\}, \quad t \in [-1, 1].$$

F is of bounded totally g -variation (11). $f \in \mathcal{F}$ is bounded, therefore $g \circ f$ is bounded, $\mathbf{I}_m(f) = C_m^g(f)$, and according to [16] (Appendix) we have the claim. \square

Corollary 1 Let $\mathbf{I}_m : \mathcal{F} \rightarrow]-1, 1[$ be a general fuzzy integral with respect to a signed \oplus -measure m , $m \in BgV$. Then

$$\mathbf{I}_m(f) = g^{-1} \left(\int g \circ f d\mu \right),$$

where integral on the right-hand side is g -integral, see [17, 18].

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