Product Type Operations between Fuzzy Numbers and their Applications in Geology

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Abstract: Multiplicative operations for fuzzy numbers raise several problems both from the theoretical and practical point of view in fuzzy arithmetic. The multiplication based on Zadeh's extension principle and its triangular and trapezoidal approximation is used in several recent works in applications in geology. Recently, new product-type operation are introduced and studied, as e.g. the cross product of fuzzy numbers and the product obtained by the best trapezoidal approximation preserving the expeted interval. We present a comparative study of the above mentioned multiplications with respect to geological applications.

Keywords: fuzzy number, cross product, trapezoidal approximation, geological resource estimation

1 Introduction

Uncertainties in different scientific areas arise mainly from the lack of human knowledge. In many practical situations the uncertainties are not of statistical type. This situation occurs mainly in the case of modeling the linguistic expressions because of their dependence on human judgement. Also, as it is shown in several recent works (see e.g. [1]) the uncertainty on different measurements due to finite resulution of measuring instruments is in many cases more possibilistic than probabilistic, since, in many applications the measurements cannot be repeated. This situation occurs mainly in Geosciences, since in this case we cannot have two holes in the same place in order to repeat the measurement so every experiment can be considered as unique. This shows us that uncertainties on the measurements in geological data are more of possibilistic type than of probabilistic type, since in order to obtain the statistical distribution of a variable we need several experiments. Fuzzy numbers allow us to modell in an easy way these non-probabilistic uncertainties. This justifies the increasing interest on theoretical and practical aspects of fuzzy arithmetic in the last years, especially directed to: operations over fuzzy numbers and properties, ranking of fuzzy numbers and canonical representation of fuzzy numbers.

Usually, the definition of addition and multiplication of fuzzy numbers are based on the extension principle ([15]). A main disadvantage of the multiplicative operation in this case is that by multiplication the shape of $L-R$ type fuzzy numbers (so triangular or trapezoidal numbers) is not preserved. In many situations this problem is solved by approximating the result of the extension principle-based multiplication by a triangular or trapezoidal number. This can lead to unexpected results in the case of iterative application of these appeoximations (i.e. can increase or decrease a defuzzified vale in a considerable way). For example in [1] the result of the product obtained by using the extension principle is approximated by a trapezoidal number considering the endpoints of the core and support.

Nevertheless, there exist other directions of development of fuzzy arithmetic. For example, in [11] new operations on fuzzy numbers are defined starting from a representation of a fuzzy number by a location index number and two fuzziness index functions.

Recently, in [2] a new multiplicative operation of product type is introduced, the so-called cross-product of fuzzy numbers and its algebraic and analytic properties are studied. The main point is that this product preserves the shape of $L-R$ fuzzy numbers under multiplication, is consistent to the classical error theory and has good algebraic and metric properties. The idea to define the cross product started from the approximation formulas ([5], p. 55) of the multiplication (obtained by Zadeh extension principle) of two $L-R$ type fuzzy numbers by an $L-R$ type fuzzy number if the spreads are small compared with the means of the numbers. The consistency of the cross product with the classical error theory is also proved in [2].

The above mentioned properties motivate to use the cross product in geological applications as a possible altenative of the product obtained by Zadeh's extension principle, mainly in the case of iterative calculations using products in each iteration.

Recently, in [9] a new method is proposed for the approximation of a fuzzy number by a trapezoidal number. This method is the best in some sense and the properties shown in [9] motivate the use of this method in order to approximate the extension priciple-based product by a trapezoidal number. This operation (regarded as a new product) can be also useful in several geological applications.

In Section 2 we recall some concepts from fuzzy arithmetic. In Section 3, we summarise the definition and some properties of the cross product. In Section 4, we discuss the method introduced in [9]. In Section 5 we propose a practical, comparative study of the previously discussed product-type operations. Applications in Geology discussed in this section concern nuclear safety assessment of a repository and the estimation of the quantity of bauxite in Halimba. At the end of the paper we present some conclusions and further research topics.

2 Basic Concepts

Let us recall the following well-known definition of a fuzzy number. The addition of fuzzy numbers and multiplication of a fuzzy number by a crisp number are provided by Zadeh's extension principle.

Definition 1 A fuzzy number is a function $u : \mathbf{R} \to [0,1]$ with the following properties:

(i) u is normal, i.e., there exists $x_0 \in \mathbf{R}$ such that $u(x_0)=1$;

(ii) $u(\lambda x+(1-\lambda)y) \geq \min\{u(x),u(y)\}, \forall x,y \in R, \forall \lambda \in [0,1]$;

(iii) u is upper semicontinuous on $\mathbf{R}$, i.e., $\forall x_0 \in \mathbf{R}$ and $\forall \varepsilon > 0$ there exists a neighborhood $V(x_0)$ such that $u(x) \leq u(x_0)+\varepsilon, \forall x \in V(x_0)$;

(iv) The set $\overline{\text{supp}(u)}$ is compact in $\mathbf{R}$, where supp $(u)=\{x \in R; u(x)>0\}$.

We denote by R_F the set of all fuzzy numbers.

Let $a,b,c \in \mathbf{R}, a<b<c$. The fuzzy number $u : \mathbf{R} \to [0,1]$ denoted by (a,b,c) and defined by $u(x)=0$ if $x \leq a$ or $x \geq c, u(x)=\frac{x-a}{b-a}$ if $x \in [a,b]$ and $u(x)=\frac{c-x}{c-b}$ if $x \in [b,c]$ is called a triangular fuzzy number.

For $0<r\leq 1$ and $u \in R_F$ we denote $[u]^r=\{x \in \mathbf{R}; u(x) \geq r\}$ and $[u]^0=\overline{\{x \in \mathbf{R}; u(x)>0\}}$. It is well-known that for each $r \in [0,1], [u]^r$ is a bounded closed interval, $[u]^r = \left[\underline{u}^r, \overline{u}^r\right]$. Let $u, v \in R_F$ and $\lambda \in \mathbf{R}$. We define the sum $u+v$ and the scalar multiplication λu by

$$[u+v]^r = [u]^r + [v]^r = \left[\underline{u}^r + \underline{v}^r, \overline{u}^r + \overline{v}^r\right]$$

and

$$[\lambda u]^r = \lambda [u]^r = \begin{cases} \left[\lambda \underline{u}^r, \lambda \overline{u}^r\right], & \text{if } \lambda \geq 0, \\ \left[\lambda \overline{u}^r, \lambda \underline{u}^r\right], & \text{if } \lambda < 0, \end{cases}$$

respectively, for every $r \in [0,1]$.

We denote by $-u = (-1)u \in R_F$ the symmetric of $u \in R_F$.

The product $u \cdot v$ of fuzzy numbers u and v, based on Zadeh's extension principle, is defined by

$$\underline{(u \cdot v)}^r = \min\{\underline{u}^r \underline{v}^r, \underline{u}^r \overline{v}^r, \overline{u}^r \underline{v}^r, \overline{u}^r \overline{v}^r\}$$
$$\overline{(u \cdot v)}^r = \max\{\underline{u}^r \underline{v}^r, \underline{u}^r \overline{v}^r, \overline{u}^r \underline{v}^r, \overline{u}^r \overline{v}^r\}.$$

Surely, the above formulas are not very practical from the computational point of view. Also, let us remark that usually the fuzzy numbers which are used in practical applications are trapezoidal. So, the requirement that a product operation should be shape-preserving seems to be natural.

For this aim, for applications in geology, the following (we will call it old) trapezoidal approximation of the product is used: the endpoints of the 0 and 1 level sets of the product determine a trapezoidal number, which is then regarded as the result of the multiplication (see [1]).

Definition 2 A fuzzy number $u \in R_F$ is said to be positive if $\underline{u}^1 \geq 0$, strict positive if $\underline{u}^1 > 0$, negative if $\overline{u}^1 \leq 0$ and strict negative if $\overline{u}^1 < 0$. We say that u and v have the same sign if they are both positive or both negative.

Let $u, v \in R_F$. We say that $u \prec v$ if $\underline{u}^r \leq \underline{v}^r$ and $\overline{u}^r \leq \overline{v}^r$ for all $r \in [0,1]$. We say that u and v are on the same side of 0 if $u \prec 0$ and $v \prec 0$ or $0 \prec u$ and $0 \prec v$.

Remark 1 If u is positive (negative) then $-u$ is negative (positive).

Definition 3 For arbitrary fuzzy numbers u and v the quantity

$$D(u,v)=\sup_{0\le r\le 1}\{\max\{|\underline{u}^r-\underline{v}^r|,|\overline{u}^r-\overline{v}^r|\}\}$$

is called the (Hausdorff) distance between u and v.

It is well-known (see e.g. [14]) that (R_F, D) is a complete metric space and D verifies $D(ku,kv)=|k|D(u,v)$, $\forall u,v\in R_F$, $\forall k\in R_F$.

3 The Cross Product

In this section we study the theoretical properties of the cross product of fuzzy numbers. Let $R_F{}^*=\{u\in R_F\ :\ u \text{ is positive or negative}\}$. Firstly we begin with a theorem which was obtained by using the stacking theorem ([12]).

Theorem 1 *If u and v are positive fuzzy numbers then $w=u\odot v$ defined by $[w]^r=\left[\underline{w}^r,\overline{w}^r\right]$, where $\underline{w}^r=\underline{u}^r\underline{v}^1+\underline{u}^1\underline{v}^r-\underline{u}^1\underline{v}^1$ and $\overline{w}^r=\overline{u}^r\overline{v}^1+\overline{u}^1\overline{v}^r-\overline{u}^1\overline{v}^1$, for every $r\in[0,1]$, is a positive fuzzy number.*

Corollary 1 *Let u and v be two fuzzy numbers.*

(i) *If u is positive and v is negative then $u\odot v=-(u\odot(-v))$ is a negative fuzzy number;*

(ii) *If u is negative and v is positive then $u\odot v=-((-u)\odot v)$ is a negative fuzzy number;*

(iii) *If u and v are negative then $u\odot v=(-u)\odot(-v)$ is a positive fuzzy number.*

Definition 4 The binary operation $\odot$ on R_F^* introduced by Theorem 1 and Corollary 1 is called cross product of fuzzy numbers.

Remark 1) The cross product is defined for any fuzzy numbers in

$R_F^{\wedge}=\{u\in R_F^*;\ \text{there exists an unique } x_0\in\mathbf{R} \text{ such that } u(x_0)=1\}$, so implicitly for any triangular fuzzy number. In fact, the cross product is defined for any fuzzy number in the sense proposed in [6] (see also [14]).

2) The below formulas of calculus can be easily proved ($r\in[0,1]$):

$$\underline{(u\odot v)}^r=\overline{u}^r\underline{v}^1+\overline{u}^1\underline{v}^r-\overline{u}^1\underline{v}^1,$$
$$\overline{(u\odot v)}^r=\underline{u}^r\overline{v}^1+\underline{u}^1\overline{v}^r-\underline{u}^1\overline{v}^1$$

if u is positive and v is negative,

$$\underline{(u\odot v)}^r=\underline{u}^r\overline{v}^1+\underline{u}^1\overline{v}^r-\underline{u}^1\overline{v}^1,$$
$$\overline{(u\odot v)}^r=\overline{u}^r\underline{v}^1+\overline{u}^1\underline{v}^r-\overline{u}^1\underline{v}^1$$

if u is negative and v is positive. In the last possibility, if u and v are negative then

$$\underline{(u\odot v)}^r=\overline{u}^r\overline{v}^1+\overline{u}^1\overline{v}^r-\overline{u}^1\overline{v}^1,$$
$$\overline{(u\odot v)}^r=\underline{u}^r\underline{v}^1+\underline{u}^1\underline{v}^r-\underline{u}^1\underline{v}^1.$$

3) The cross product extends the scalar multiplication of fuzzy numbers. Indeed, if one of operands is the real number k identified with its characteristic function then $\underline{k}^r=\overline{k}^r=k, \forall r\in[0,1]$ and following the above formulas of calculus we get the result.

The main algebraic properties of the cross product are the following.

Theorem 2 *If* $u,v,w\in R_F^*$ *then*

(i) $(-u)\odot v=u\odot(-v)=-(u\odot v)$;

(ii) $u\odot v=v\odot u$;

(iii) $(u\odot v)\odot w=u\odot(v\odot w)$;

(iv) *If* u *and* v *have the same sign then* $(u+v)\odot w=(u\odot w)+(v\odot w)$;

(v) $(u\odot v)^{\odot n}=u^{\odot n}\odot v^{\odot n}, \forall n\in N^*$ *, where* $a^{\odot n}=\underbrace{a\odot\ldots\odot a}_{n \text{ times}}$ *for any* $a\in R_F^*$.

Remark 1) If u is positive and v negative (or u is negative and v positive) then the property of distributivity in (iv) is not verified even if u and v are real numbers.

2) The above properties (i)-(iii) hold for the usual product " $\cdot$ " based on the extension principle. The property (iv) holds in a weaker form: If u and v are on the same side of 0 then for any $w, w\prec 0$ or $0\prec w$ we have $(u+v)\cdot w=(u\cdot w)+(v\cdot w)$.

The so-called L - R fuzzy numbers are considered important in fuzzy arithmetic. These and their particular cases triangular and trapezoidal fuzzy numbers are used almost exclusively in applications.

Definition 5 ([5], p. 54, [14]) Let $L,R : [0,+\infty) \to [0,1]$ be two continuous, decreasing functions fulfilling $L(0) = R(0) = 1, L(1) = R(1) = 0$, invertible on $[0,1]$. Moreover, let a^1 be any real number and suppose $\underline{a},\overline{a}$ be positive numbers. The fuzzy set $u : \mathbf{R} \to [0,1]$ is an L - R fuzzy number if

$$u(t) = \begin{cases} L\left(\frac{a^1-t}{\underline{a}}\right), & \text{for } t \le a^1 \\ R\left(\frac{t-a^1}{\overline{a}}\right), & \text{for } t > a^1. \end{cases}$$

Symbolically, we write $u = \left(a^1, \underline{a}, \overline{a}\right)_{L,R}$, where a^1 is called the mean value of u, $\underline{a},\overline{a}$ are called the left and the right spread. If u is an L - R fuzzy number then (see e. g. [13])

$$[u]^r = \left[a^1 - L^{-1}(r)\underline{a}, a^1 + R^{-1}(r)\overline{a}\right].$$

Theorem 3 *If u and v are strict positive L - R fuzzy numbers then $u \odot v$ is a strict positive L - R fuzzy number.*

Since we are interested mainly in the applications of the cross product we may restrict our attention to positive fuzzy numbers, however in other cases some similar properties can be obtained (see [3]).

The cross product verifies the following metric property.

Theorem 4 *If u,v have the same sign and $w \in R_F^*$ then*

$$D(w \odot u, w \odot v) \le K_w D(u,v),$$

where $K_w = \max\{\left|\overline{w}^1\right|, \left|\underline{w}^1\right|\} + \overline{w}^0 - \underline{w}^0.$

By using the previous metric property, several properties can be obtained with respect to continuity, differentiability (using the H-differential) and integrability of the product of fuzzy-number-valued functions (see [2]).

The following interpretation related to error theory is a further theoretical motivation of the use of the cross product of fuzzy numbers. Indeed, the consistency of the cross product with the classical theory motivates its use in the case of modelling uncertain data (uncertainty being due to errors of measurement).

We introduce two kinds of errors of fuzzy numbers corresponding to absolute error and relative error in classical error theory and we study these with respect to sum and cross product.

Definition 6 Let u be a fuzzy number. The crisp number $\Delta_L^r(u) = \underline{u}^1 - \underline{u}^r$ is called r - error to left of u and the crisp number $\Delta_R^r(u) = \overline{u}^r - \overline{u}^1$ is called r - error to right of u, where $r \in [0,1]$. The sum $\Delta^r(u) = \Delta_L^r(u) + \Delta_R^r(u)$ is called r -error of u.

If u expresses the fuzzy concept A then $\Delta_L^r(u)$ and $\Delta_R^r(u)$ can be interpreted as the values of tolerance of level r from the concept A to left and to right, respectively. For example, if the triangular fuzzy number $u = (5,7,9)$ expresses "early morning" then $\Delta_L^{\frac{1}{2}}(u) = 1$ (one hour) is the tolerance of level $\frac{1}{2}$ of u towards night from the concept of "early morning" and $\Delta_R^{\frac{1}{4}}(u) = 0.5$ (30 minutes) is the tolerance of level $\frac{1}{4}$ of u towards moon from the concept of "early morning".

A new argument in the use of addition of fuzzy numbers as extension (by Zadeh's principle) of real addition is the validity of the formula

$$\Delta^r\left(u+v\right) = \Delta^r\left(u\right) + \Delta^r(v)$$

which is consistent to the classical error theory. It is an immediate consequence of the obvious formulas

$$\Delta_L^r\left(u+v\right) = \Delta_L^r\left(u\right) + \Delta_L^r\left(v\right)$$

and

$$\Delta_R^r\left(u+v\right) = \Delta_R^r(u) + \Delta_R^r\left(v\right).$$

Now, let us study the relative error of the cross product.

Definition 7 Let u be a fuzzy number such that $\underline{u}^1 \neq 0$ and $\overline{u}^1 \neq 0$. The crisp numbers $\delta_L^r\left(u\right) = \frac{\Delta_L^r(u)}{\left|\underline{u}^1\right|}$ and $\delta_R^r\left(u\right) = \frac{\Delta_R^r(u)}{\left|\overline{u}^1\right|}$ are called relative r -errors of u to left and to right. The quantity $\delta^r(u) = \delta_L^r(u) + \delta_R^r(u)$ is called relative r -error of u.

Theorem 5 *If u and v are strict positive or strict negative fuzzy numbers then*

$$\delta^r\left(u \odot v\right) = \delta^r\left(u\right) + \delta^r\left(v\right).$$

Corollary 2 *If* u *is a strict positive fuzzy number then* $\delta_L^r\left(u^{\odot n}\right)=n\delta_L^r\left(u\right)$, $\delta_R^r\left(u^{\odot n}\right)=n\delta_R^r\left(u\right)$ *and* $\delta^r\left(u^{\odot n}\right)=n\delta^r\left(u\right)$.

The above theorems show us that the cross product is consistent with the classical error theory (the propagation of errors is governed by a similar law as in the classical case).

4 New Trapezoidal Approximation of the Product

As we have discussed in the previous sections, the usual (Zadeh's extension principle-based) product do not preserve the shape of the operands. Thus, the result of the product, for computational purposes has to be approximated by a trapezoidal number. A new, axiomatic approach to this problem has been introduced in [9]. We will call the trapezoidal approximation of the product based on this method as new trapezoidal approximation of the product.

The method proposed in [9] gives the best approximation of the product under some appropiate conditions. These conditions are natural, so in the approximation of the product the result obtained is motivated from the theoretical point of view. Let us regard the trapezoidal approximation as an operator T, $T : R_F \to R_F$, which for a given fuzzy number u gives its trapezoidal approximation. The list of requirements which have to be satisfied by an operator of this type are given in [9] below.

1 Conserving some fixed α -cut. E.g. if the operator preserves the 0 and 1 level sets, then the old trapezoidal approximation is reobtained.

2 Invariance to translation of the operator T.

3 Invariance with respect to rescaling.

4 Monotonicity with respect to inclusion.

5 Idempotency (i.e. the trapezoidal approximation of a trapezoidal number is itself).

6 To be best approximation, that is, it should be the nearest in some prescribed sence ($D(T(u),u)\le D(x,u)$ for any trapezoidal number x).

7 Conserves the so-called expected interval, that is the original fuzzy number and its approximation have the same expected interval (Let us recall here that the expected interval of $u\in R_F$ is $\left[\int_0^1\underline{u}^r dr,\int_0^1\overline{u}^r dr\right]$.

8 Continuity.

9 Compatiblity with the extension principle.

10 Monotonicity with respect to some ordering between fzzy numbers.

11 Invariance with respect to correlation (see [7]).

In [9] the authors propose a trapezoidal approximation which is best approximation and preserves the expected interval, that is conditions 6 and 7 are required. In this case, for $u \in R_F$ we obtain the trapezodal fuzzy number (t_1, t_2, t_3, t_4), where

$$t_1 = -6\int_0^1 r\underline{u}^r\,dr + 4\int_0^1 \underline{u}^r\,dr,$$
$$t_2 = 6\int_0^1 r\underline{u}^r\,dr - 2\int_0^1 \underline{u}^r\,dr,$$
$$t_3 = 6\int_0^1 r\overline{u}^r\,dr - 2\int_0^1 \overline{u}^r\,dr,$$
$$t_4 = -6\int_0^1 r\overline{u}^r\,dr + 4\int_0^1 \overline{u}^r\,dr.$$

In [9], the authors proved also that conditions 2,3,4,5,8,9,10,11 are fulfiled. Let us remark also, that the expected value of a fuzzy number in the sense of [hei92] (this is called in several works defuzzification by center of area method) is the same as the expected value of its trapezoidal approximation. These results lead us to the conclusion that the approximation method proposed in [9] can be useful for applications in geology.

5 Applications of the Product-type Operations in Geology

Recently, fuzzy arithmetic has found several applications in geology (see [1]). In the above cited work the usual (Zadeh's extension principle based) product is used in applications concerning nuclear safety assessment of a repository and for estimation of resources of solid mineral deposits. In this section we propose an alternative study of the same problems, by using the cross product and the new trapezoidal approximation.

The reasons of the possible usefulness of the cross product are the followings. Firstly, in this case the shape of the result of the product is conserved, i.e. the product of triangular numbers is triangular and the product of trapezoidal numbers is trapezoidal. Secondly, the 1-level sets are better taken into account by the use of cross product. Also, the consistency of the cross product with the classical error theory motivate this study.

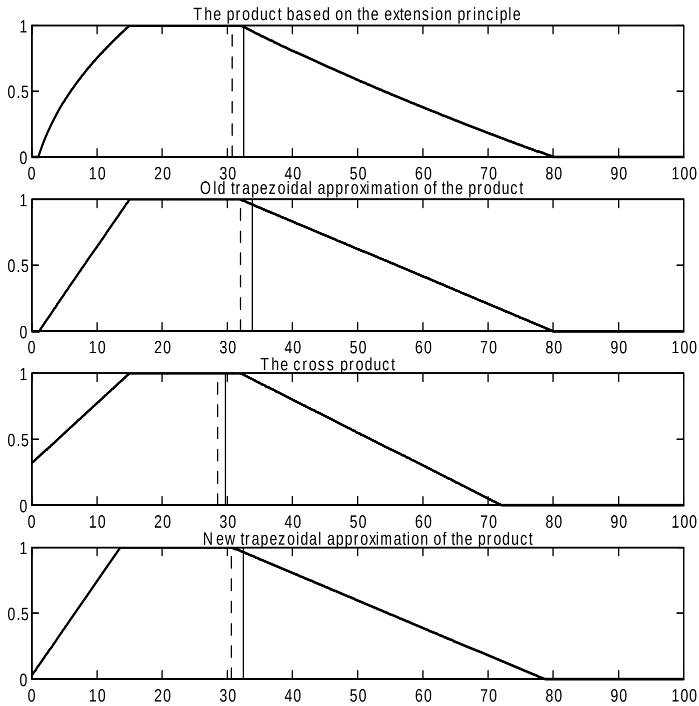

Figure 1

The product of two trapezoidal fuzzy numbers obtained by the different approaches discussed in the paper

The reasons for the possible usefulness of the new trapezoidal approximation are the properties presented in the previous section, that is it preserves the expected interval (also the expected value) and it is the best approximation in some sense.

The product-type operations between fuzzy numbers appear also in several new research fields of geology, such as safety assessment of spent nuclear fuel repositories. The fuzzy modell of the repository is given by a system of linear, coupled fuzzy differential equations. The initial values of several variables and also several coefficients are subject of non-statistical uncertainty, i.e. these are fuzzy numbers. The product-type operations appear in this case in the equations in several places. Firstly, some coefficients in the equations are multiplied by a fuzzy valued function. Also, in the solution of the system obtained by using the extension principle, several times we have to compute powers and even exponentials of a fuzzy number. The three above discussed possible ways to perform multiplication of two fuzzy numbers are illustrated in Figure 1 for the product of the trapezoidal fuzzy numbers $(1,5,8,10)$, $(1,3,4,8)$. The solid vertical line and the dashed vertical line represent the defuzzified values of the results by centroid and expected values (center of area) methods respectively.

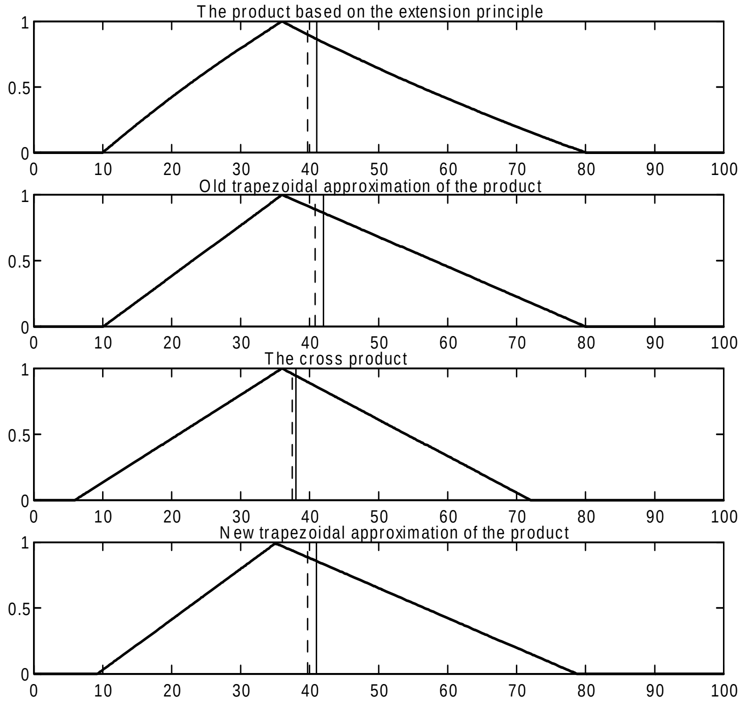

Figure 2
The product of two triangular numbers

In Figure 2, the results of the product of two triangular numbers, $(2,6,8)$, $(5,6,10)$ are presented.

The Figures 1 and 2 do not show a striking difference between the results of the different methods. The difference can be significant if we perform iterative computations with the fuzzy numbers.

In order to show this we consider the exponential functions obtained as power series with respect to the product type operations discussed above. In Figure 3 the results of the exponential-type functions are presented. The fuzzy number considered in the exponent is $(2.2, 4.6, 4.7, 5)$. Solid thin line represent again defuzzification by centroid method, while dashed line the expected value.

Significant difference can be observed between the different results in this case, that is indeed, the iterative use of the product operations leads to different result even after defuzzification. This problem can be avoided by considering and examining all the operations in all the practical problems considered and taking them into account in risk analysis.

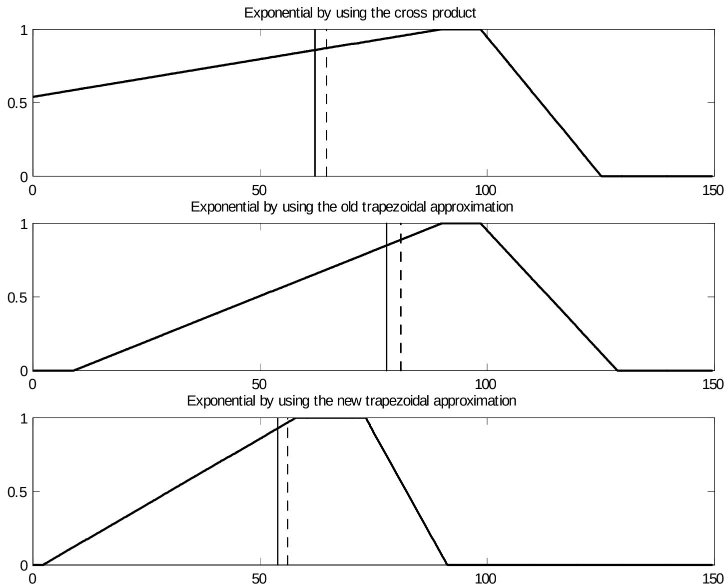

Figure 3
The exponential of a fuzzy number

This figure suggests us also, to be careful with the use of the cross product in the construction of an exponential, since in some cases the result given by this operation is negative, fact which is impossible from a possibilistic point of view. The figure suggests that probably the best behaviour is that of the new trapezoidal approxmation. Unfortunately, since its very high computational complexity this method cannot be effectively used for modeling purposes in nuclear safety assessment.

The next application presents as in [1], resource estimation on several bauxite deposites in Hungary. In the same way as with the traditional methods, the tonnage of the resources is obtained by the product of the deposit area, the average thickness and the average bulk-density of the studied ore or mineral commodity (see Figure 4 and see also [1]). Large deposits can be split into blocks, preferably along natural boundaries, such as tectonic lines.

We present the results obtained by the old trapezoidal approximation, the results obtained by using the cross product and the new trapezoidal approximation compared with the product based on the extension principle (see Figures 5, 6).

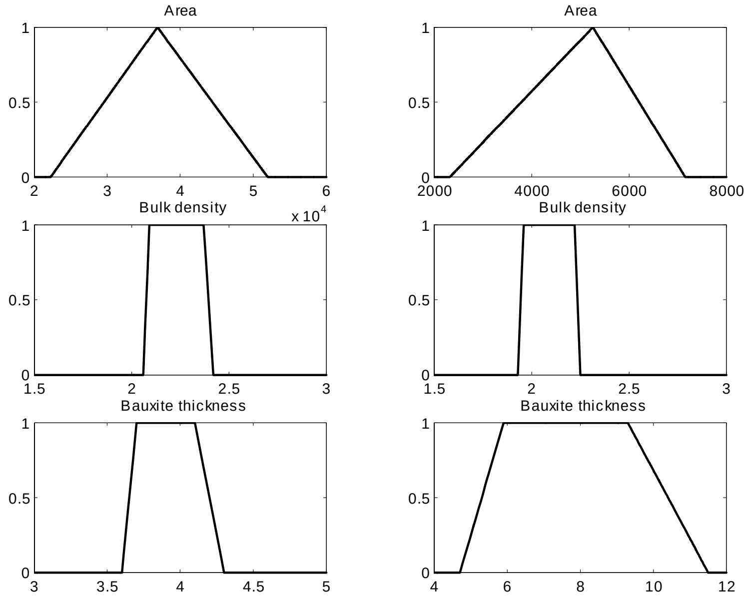

Figure 4
Fuzzy numbers used for the resource calculations at Sőcz-Szárhegy I and I/A (left) and Óbarok IX (right) sites

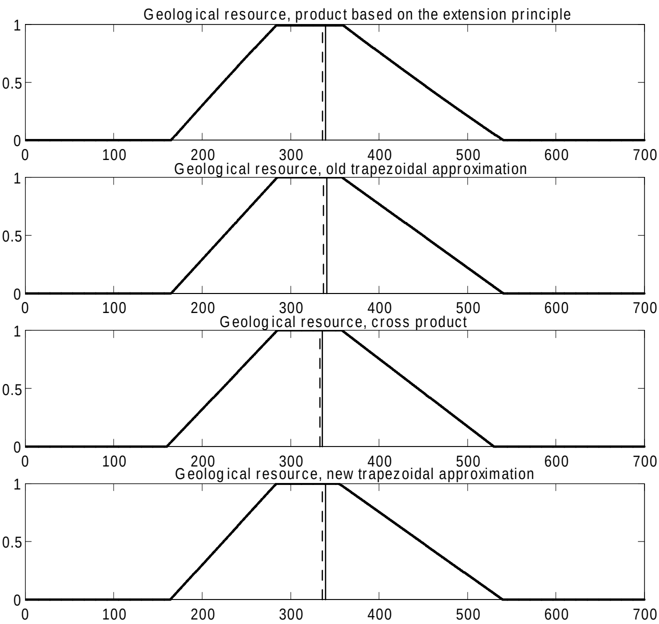

Figure 5
Fuzzy numbers of the tonnage calculation, Szőc-Szárhegy I. and I/A. (thousand tones)

We observe that, if we defuzzify the results obtained by the three different product-type operations we conclude that the results are different. Also, we observe that after defuzzification (by centroid method) the result of the cross product in the study of the deposites at both sites is smaller than that of the new trapezoidal approximation, which is smaller at its turn than the old trapezoidal approximation of the product. So, the risks of an investment at this site can be more realistically evaluated if we use the cross product or the new trapezoidal approximation, however the results are not very different. From the risk analysis point of view of the investment, the most important information is not in the defuzzified value, but in the trapezoidal fuzzy numbers themselves. Indeed, from a possibilistic point of view, the fuzzy numbers contain much more information than only the defuzzified vales and the risks can be better evaluated considering the tonnage as a fuzzy number.

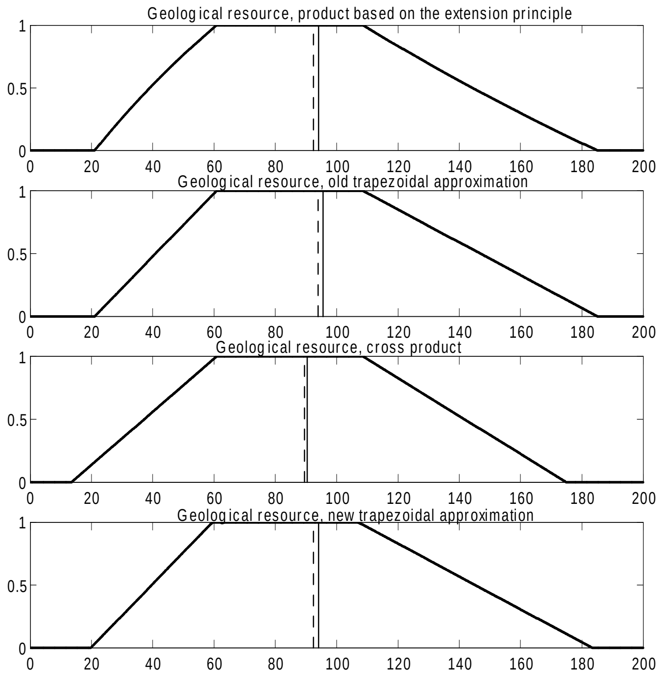

Figure 6

Fuzzy numbers of the tonnage calculation, Óbarok IX. (thousand tones)

Conclusions and Further Research

Several methods for the multiplication of fuzzy numbers are discussed from the theoretical and practical point of view. As a conclusion of this research we can state that the theretical properties of the cross-product and the new trapezoidal approximation motivate the usefulness of both methods in geology, however, usually the most conservative method is the old trapezoidal approximation. So, there exist reasons for using all the above mentioned approaches and to take into account the results of all the approaches in a risk analisys or a safety assessment.

From the computational point of view, let us remark that the old trapezoidal approximation and the cross product can be computed in an easy way taking into account only the endpoints of the core and support of the trapezoidal operands and the extension principle can be avoided in these cases. In the case of the new trapezoidal approximation, since the implementation of this operation involves the use of the extension principle and then numerical integration on the side-functions of the fuzzy numbers, the computational complexity is high. This makes almost impossible to use the new trapezoidal approximation in iterative computations.

For further research we propose effective implementation of the new trapezoidal approximation, and the design of new, computationally simple methods for the approximation of the product based on the extension principle.

References

[1] Gy. Bárdossy, J. Fodor, *Evaluation of Uncertainties and Risks in Geology*, Springer Verlag, Berlin Heidelberg, New York, 2004

[2] A. I. Ban, B. Bede, Cross product of fuzzy numbers and properties, *Journal of Fuzzy Mathematics*, to appear

[3] A. I. Ban, B. Bede, Cross product of $L-R$ fuzzy numbers and properties, *Anals of Oradea Univ., Fasc. Matem.* 9(2003), 95-108

[4] M. Delgado, M. A. Vila and W. Voxman, On a canonical representation of fuzzy numbers, *Fuzzy Sets and Systems*, 93(1998), 125-135

[5] D. Dubois and H. Prade, *Fuzzy Sets and Systems*, Academic Press, New York, 1980

[6] R. Fuller and T. Keresztfalvi, On generalization of Nguyen's theorem, *Fuzzy Sets and Systems*, 41(1991), 371-374

[7] R. Fullér, P. Majlender, On interactive fuzzy numbers, *Fuzzy Sets and Systems* 143(2004) 355-369

[8] R. Goetschel and W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets and Systems*, 18(1986), 31-43

[9] P. Grzegorzewski, E. Mrówka, Trapezoidal approximations of fuzzy numbers, *Fuzzy Sts and Systems*, 153(2005), 115-135

[10] S. Heilpern, The expected value of a fuzzy number, *Fuzzy Sets and Systems*, 47(1992), 81-86

[11] Ming Ma, M. Friedman and A. Kandel, A new fuzzy arithmetic, *Fuzzy Sets and Systems*, 108(1999), 83-90

[12] M. L. Puri and D. A. Ralescu, Differentials of fuzzy functions, *J. Math. Anal. Appl.*, 91(1983), 552-558

[13] M. Wagenknecht, R. Hampel and V. Schneider, Computational aspects of fuzzy arithmetics based on Archimedean t-norms, *Fuzzy Sets and Systems*, 123(2001), 49-62

[14] Congxin Wu and Zengtai Gong, On Henstock integral of fuzzy number valued functions (I), *Fuzzy Sets and Systems*, 120(2001), 523-532

[15] L. A. Zadeh, Fuzzy sets, *Inform. and Control*, 8(1965), 338-353