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Commemoration of the UNESCO Bolyai Memorial Year

Róbert Oláh-Gál,
Sapientia Hungarian University of Transylvania, Faculty of Economics and Socio-Human Sciences and Engineering
olahgalrobert@uni.sapientia.ro

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Abstract

As it was emphasized in the author's several theses and books the Bolyai-Lobachevsky geometry proved not to be valid in the 3-dimensional Euclidean space. Lately, it was mentioned in some notes in Targu-Mures (Romania) that the Heureka program (Duna TV Budapest) reported about the worldwide first implementation of the Bolyai-Lobachevsky's geometry in architecture [4]. Without doubting the magnificence of the concerning buildings or the artistic or architectural value of some churches, these allegations must definitely be denied.

Keywords: Bolyai, Bolyai-Lobachevsky geometry, pseudosphere, Hilbert

Figure 1: The Greek-Catholic Church "The Assumption of the Mother of God" in Reghin Romania (architect Gerő Szekeres)

A well-informed differential geometer immediately notices this, because 100 years ago David Hilbert proved that Bolyai geometry is not realized in 3-dimensional Euclidean space as an

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embedded geometry[1, 2, 3]. Embedded means exactly what the architect does; to realize buildings in 3-dimensional Euclidean space.

So what about the pseudosphere? Around any point of the pseudosphere, a very small environment (small piece of surface) can be considered, on which the Bolyai geometry prevails locally (in a small way). The entire surface of the pseudosphere is finite, while the Bolyai-Lobachevsky plane is infinite, and its geodesics [Figure 2., 5., 6.], i.e. the Bolyai straight lines, twist onto the conical part and two geodesics intersect at many points (apart from the meridian curves, which are specific geodesics) [5, 6].

Figure 2: On the pseudosurface, the straight lines, that is, the geodesics, wind up on the conical part. It clearly illustrates that it is a local model and models only a finite part of the hyperbolic plane.

Figure 3: A pseudosphere with a non-Euclidean line, i.e. a geodesic. It can be seen that the straight line is self-intersecting.

2023. Vol. 13. No. 26. ISSN 2062-2597
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For example, constructing a spherical surface does not mean using elliptical geometry. A sphere is essentially a Euclidean object. It is a different matter that Gauss proved in his famous work and in his so-called miraculous theorem that surfaces with constant curvature have an independent internal geometry. But we also know that elliptical geometry is not realized globally on the spherical surface either, because two principal circles (two elliptical lines = geodesic lines) intersect at 2 opposite points, even though in all 3 classical geometries (Euclidean, hyperbolic and elliptical) two lines are can intersect at a single point.

If someone comes up with a gadget that contains a right-angled triangle in which the Pythagorean Theorem is not true, we will surely be amazed! Because we know that there is no right-angled triangle in which the Pythagorean Theorem is not true. Well, there is no such building that implements Bolyai-Lobachevsky geometry, at least in 3-dimensional Euclidean space. For now, all we know is that it can exist in 5 or 6 dimensional space [7]. (Based on Danilo Balusa's embedding theorem [1].)

In architecture, it is generally important to be able to make some kind of formwork (moulding) for the roof structure or curved building piece that requires a concrete structure. Geometrically, this means that the surface is approximated with a line surface, since most of the time the formwork is made of pieces of boards, that is, we cover the surface with a line surface. So what we do is to "straighten" the curvature in parts (this is one of the most important techniques of mathematical analysis) [Figure 3.].

If a roof structure is covered with straight sections, it has already become a linear surface. And this means that the curvature of that surface is not constant negative, so we are already further away from the Bolyai geometry. Such a technique has been used in many places so far, such as for hyperboloids of rotation, such as the cold storages at the former Fântânele Thermal Power Plant [Figure 4.].

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Figure 4: The hyperboloid with a canvas (hyperboloid of rotation). It is a ruled surface with zero Gauss curvature.

Fântânele (Romania) Thermal Power Plant.

The roof structure of the Predeal Railway Station is also such a parabolic hyperboloid, which is also a linear surface.

Of course, the geometry of János Bolyai may have given the architect the idea, the roof structure of the building may be based on the shape of the pseudosphere, but that does not mean that he will be the first in the world to use Bolyai's geometry in architecture. The pseudosphere itself is a Euclidean figure. Gauss proved in 1827 that the internal geometry of a surface means a new geometry, as I have already mentioned. János Bolyai did not know this because he did not read Gauss's "Disquisitiones generales circa superficies curvas". (It is true that János Bolyai ordered this work written by Gauss founding differential geometry, from the bookseller Thierry in Sibiu, but the bookseller was unable to acquire it.) But Eugenio Beltrami proved in 1868 that Bolyai's geometry applies locally on the pseudosphere [Figure 3., 5., 6.]. By the way, the pseudosphere was already known 50 years before the birth of János Bolyai.

David Hilbert, one of the greatest mathematicians of the 20th century, proved that in 3-dimensional Euclidean space, Bolyai-Lobachevsky geometry, like the internal geometry of a

2023. Vol. 13. No. 26.
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surface, is not realized. This is a mathematical theorem meaning that it is absolutely valid. There are no exceptions in mathematics. A single counterexample already invalidates a statement, which is why we talk about conjecture until then. Not even an extraterrestrial being can come up ten thousand years from now that he found a gadget that implemented Bolyai-Lobachevsky geometry in 3-dimensional Euclidean space. The idea will still be laughed at. Mathematics can formulate a priori (prior to all experiences) synthetic, eternally valid, absolute judgments that have nothing to do with this material world, or shadow world, if you like. This is its strength and essence!

Unfortunately, no architect in the world will be able to build the building that would realize the Bolyai geometry. It is simply not possible because such a mathematical object does not exist in 3-dimensional space!

About Bolyai-Lobachevsky's geometry, with a physicist’s eye

Perhaps in the clutches, Nobel Prize-winning physicist Stephen Weinberg formulated the essence of Bolyai-Lobachevsky's geometry when he said that Bolyai- Lobachevsky’s geometry does not fit into our 3-dimensional world, which is our experience.

Indeed, this is the reason why we know little about the Bolyai geometry and we have so many difficulties to illustrate it. The Bolyai geometry is only understood by professionals even today. Today, 180 years after its creation, a lot of inaccurate statements are made about it. For example, we encounter such things: "he Bolyai geometry, where the parallels meet," or "the Bolyai geometry is realized on the Pseudosphere" or "an architect built a church on the basis of the Bolyai geometry in the world" and so on. These statements are wrong!

In our paper, we want to explain why the Bolyai geometry does not fit into the 3-dimensional space.

Pseudosphere

The Bolyai geometry is realized locally (small) on the pseudosphere (on the surface formed by the traction curve). What does it mean to be local (small)? It means that it prevails in one point and in its very small environment. The pseudosphere has a constant negative curvature, there are some of this type of surfaces in the 3-dimension, but they are not complete.

2023. Vol. 13. No. 26. ISSN 2062-2597
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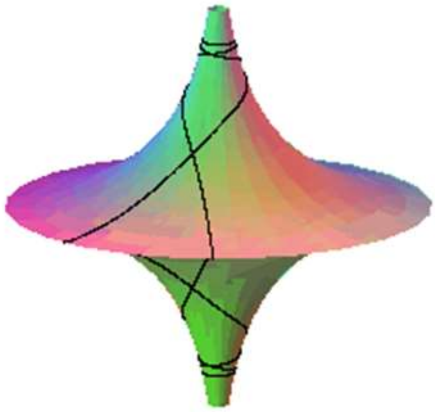

Figure 5: The pseudosphere with geodetics

The fact that a surface is incomplete means that, for example, the geodetic lines corresponding to the lines are, for example, finite according to the internal geometry of the surface. So straight lines from the straight lines are realized. The geodetic one is the shortest distance between two points. For this reason, geodesics can be called even the shortest lines on the surface (For example, on the ball surface, the circle of circles, the circles whose plane passes through the centre of the sphere, are geodetic on the cylinder, the sculptors, the creators, and the perpendicular circles of the creators). A surface would be complete if we could start geodesic from any point and in any direction, and can be continuously extended so there is no end. Well, with the pseudosphere and all the other constants with negative curvature of three dimensional surfaces, the problem is that the geodetics are finite. On the pseudosphere as a rotation surface merely the meridians are endless, the other geodetics are of finite lengths. (The meridian curve is also geodetic on each rotation surface. On a rotating surface, the curves obtained by intersecting a plane with the axis of rotation are called meridian curves, so rotation of the meridian curve causes the rotation surface). In the pseudosphere, the geodesics roll over to the cone, they turn around and end in the basic circle. (See figure 2). Thus, it does not fulfil the basic truth to be infinitely extendable in any direction, nor does it fit the fact that two different geodetics have only one common point (i.e. two intersecting straight lines have only one common point).

2023. Vol. 13. No. 26.
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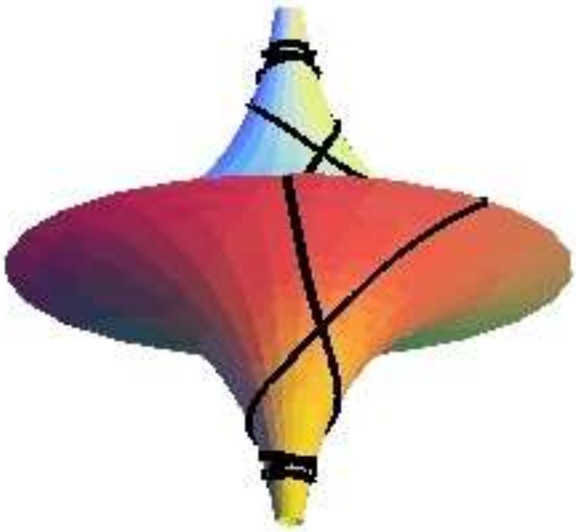

Figure 6: In the pseudosphere, the geodesics roll over to the cone, they turn around and end in the basic circle.

That is why it is necessary to exit the 3-dimensional space and examine whether it is possible to find a surface that correctly models the Bolyai-Lobachevsky geometry in all respects.

What does it mean to quit the dimension?

Take a one-dimensional shape: This is the straight line (for example, the real axis of the shaft, which models time in classical physics). It is an infinite manifold that can be moved in itself. If a section is cut out of it and the two ends of the section are glued together, the circle is still a one-dimensional multitude (because it can be given by a parameter) but the adhesion was already in the 2-dimensional space. So, the circle is embedded in the 2-dimensional space, the circle does not fit in the one-dimensional space, although it is a one-dimensional manifold.

Take a strip of paper (a rectangular flat piece) and glue it along its short side. A cylinder emerges from it. It is clear that the 2-dimensional space had to be removed and the adhesion was done in the 3-dimensional space. If, on the other hand, we wrap one over the paper strip and bond it, the Möbius band is formed, which is a one-sided surface. Now take the cylinder and stick it together at the two rounds. Thus the torus is created in the 3-dimensional space. But if the roller is rolled and glued together like a paper strip, we can only do this roll in Dimension 4, so the so-

2023. Vol. 13. No. 26. ISSN 2062-2597
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called Klein cannon is created. This is a one-sided closed surface in 4-dimensional space. That's why you have to quit the 3rd dimension!

There are some facts worth mentioning here. When we enter Dimension 4, a physicist with 4 dimensional vision will see what's in our heads. That is to say, in our skull (of course in our bodies). The circle divides the plane in the plane into two disjoint (uncompressed) ranges. But the circle in the 3-dimension does not divide the two disjointed parts of the space. The sphere divides the space into two disjoint ranges, but the sphere does not divide the 4-dimensional space into two disjoint ranges. So a physician with 4-dimensional vision does not have to break our skull, he can perform brain surgery from the fourth dimension.

In the 3-dimensional space, the sphere is the only prototype on a constant positive curvature surface. This means that if a surface is full and its curvature is $\frac{1}{r^2}$ Constant, then the radius is the radius r. And, if two surfaces have a constant positive curve with the same constant value, the two surfaces can be embedded in one motion (i.e. essentially the two surfaces are the same). This is called the positive constant curved surfaces in the 3-dimensional stiffness and uniqueness. Well, this fact is not true in the 4-dimensional space. There are infinite many "two-dimensional" curvature "spheres" with the same curvature value that cannot be covered by a motion. The discrepancy is as follows: the sphere is a rotation surface when rotating the circle around an axis (the circle is rotated about its diameter). If the shaft is "bent" in dimension 4, everything is preserved, but the bending of the axis of rotation can be done in many different ways. So we get infinitely many spheres. (Note the 4-dimensional space between the 2-dimensional sphere and the hypersphere. The hypersphere is already a 3-dimensional manifold, which we have not discussed here!)

What is János Bolyai's new, different world?

János Bolyai's lines from one of his letters come to my mind: *" I created a new, different world from nothing.*" - we can scarcely know anything about this new, different world. In a few lines, we try to outline János Bolyai’s new, different world.

If someone says that, for example, a certain chair, object or building was made based on the Bolyai geometry, that statement is not true! In theory, that would be on a par with someone saying he found a rectangular triangle in which the Pythagorean Theorem was not true, or found the perpetual motion, or made gold from some alloy. But if someone says that in the atomic structure he has discovered an effect, a chemical bond, a physical or a chemical process that follows the Bolyai-Lobachevsky geometry, that can be true. In quantum mechanics the Bolyai-Lobachevsky geometry can be realized!

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The mathematical foundation of quantum mechanics is based on John von Neumannás name, who considered the Hilbert spaces to be the most suitable ones. Remember that today we use mobile phones thanks to quantum mechanics, which was first described by John von Neumann as axiomatized by quantum mechanics. David Hilbert (1862-1943) proved in 1901 that the Bolyai plane geometry does not prevail in the 3-dimensional Euclidean space as an embedded geometry. To honour David Hilbert’s memory, geometers introduced the Hilbert spaces. This is a natural generalization of the Euclidean spaces. Well in the Hilbert space, Bolyai-Lobachevsky's geometry already exists as an embedded geometry. This was proved by the German mathematician Ludwig Bieberbach (1886-1982) in 1932. In 1955, Danilo Blanuša (1903-1987), a Croatian mathematician, proved that Bolyai-Lobachevsky's flat geometry can be embedded in the 6-dimensional space. So the Bolyai geometry fits in the 6-dimensional space, but not in the 3-dimensional one.

Physically, embedded geometry means the inner geometry of an existing, but non-identical object in the surrounding world. As in a common space, for example, there is a ball. Spherical geometry is the property of others, like the surrounding world in which it exists.

There is no similarity in the Bolyai geometry. That means there is no way to shoot, film, and architects could not make a plan. They would only have the same "figurines", they cannot be smaller, not proportional, etc. The child would not look like a parent and would not grow because he was just as old as the parent.

In the Bolyai geometry, the geometrical location of the points equidistant from a line is no longer a straight line but a hypercycle.

On the Bolyai Square, schoolchildren would have their own skip and hypercycles, and they would have their rulers and Paracycles.

Circumcision and hypercycles are tools that can be used to draw a variety of regular, self-moving "lines", a (dimensional geometric multitude), but they are not all that coincide. The definition of the circle in the Bolyai geometry is the same as in the Euclidean, that is, the geometrical location of the points that are at a distance from a fixed point.

The ruler and Paracycle-clamp are devices that produce a completely straightforward, self-moving line that is all-in-one. Two such line items can always be folded with one move. The definition of the Paracycle is a bit more complicated: since in the Bolyai geometry there exist one-way, cohesive, parallel lines, the perpendicular line (trajectory) is called a Paracycle. Thus, the Paracycle is the perpendicular (orthogonal) trajectory of the longitudinal centre of gravity parallel to each other. Therefore, should it be worth considering thoroughly whether elemental particles follow such pathways?

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ISSN
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The Bolyai geometry is endless, just like the Euclidean. The Euclidean plane and the hyperbolic plane are diffeomorphic, which means they can be continuously interlinked. This cannot be achieved with the elliptical plane (such as the surface of the sphere) because it is finite! Thus in the Bolyai geometry straight lines are infinite!

In the Bolyai geometry, a triangle area is maximum π if its sides are endless. The area of a triangle depends only on the angles, and the area of the triangle with which all angles are zero (the area is π).

To summarize the point, I think it would be right if Stephen Weinberg's (a Nobel Prize-winning physicist) claim pervaded public opinion: the Bolyai geometry does not fit into the 3-dimensional space.

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